Degrees of Freedom of the Rank-Deficient Interference Channel With Feedback

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***Abstract*— We study the sum degrees of freedom (DoFs) of the K-user rank-deficient interference channel with feedback. For the two-user case, we characterize the sum DoF by developing an achievable scheme and deriving a matching upper bound. For the three-user case, we develop a new achievable scheme which employs interference alignment to efficiently utilize the dimension of the received signal space. In addition, we derive an upper bound for the general K-user case and show the tightness of the bound when the number of antennas at each node is sufficiently large. As a consequence of these results, we show that feedback can increase the DoF when the number of antennas at each node is large enough as compared with the ranks of channel matrices. This finding is in contrast to the full-rank interference channel where feedback provides no DoF gain. The gain comes from using feedback to provide alternative signal paths, thereby effectively increasing the ranks of desired channel matrices.**

***Index Terms*— Degrees of freedom, feedback, interference alignment, interference channel, rank-deficient channel.**

I. INTRODUCTION

IT IS well known that feedback cannot increase the capacity of memoryless point-to-point channels [1]. Although the capacity of multiple access channels can in fact increase when feedback is present, the gain is bounded by one bit for the Gaussian case [2]. These results give a pessimistic view on feedback capacity, although feedback can still be useful for simplifying coding strategies as well as improving reliability [3]. Recent work [4], however, has shown that in interference channels, feedback can provide more significant gains. Specifically, it is shown that the capacity gain due to feedback becomes arbitrarily large for certain channel parameters (unbounded gain). The gain comes from the fact that feedback can help efficient resource sharing between the interfering users. In the process of deriving this conclusion, Suh and Tse [4] has characterized the feedback capacity region to within 2 bits of the two-user Gaussian interference channel.

The results of [4] indicate that feedback enables a significant capacity improvement of multi-user networks with interfering links. However, if we turn our attention to degrees of freedom (DoF), feedback fails to provide promising results. From the results of [5] and [6], it has been shown that feedback cannot improve the sum DoF for the two-user full-rank Gaussian MIMO interference channel.[1] Therefore, feedback can provide unbounded capacity gain but cannot increase the DoF in the full-rank channel.

In this work, we show that feedback, however, can increase the sum DoF in the *rank-deficient* interference channel. The rank-deficient channel captures a poor scattering environment where there are only few signal paths between nodes. For example, for rooftop-to-rooftop communications in which transmit and receive antennas are mounted high above the ground, the angular spread becomes very low [8]–[11], and, as a result, the channel matrix becomes rank deficient due to the lack of multipath. In addition, for massive MIMO communications in which each node has numerous antennas, channel matrices cannot have full rank unless there are enough number of signal paths between nodes. The non-feedback DoF of the rank-deficient interference channel has been studied in [12] and [13], and the optimal DoFs for the two-user and three-user cases have been established in [13]. In this paper, we now investigate the effects of *feedback* on the sum DoF of the rank-deficient interference channel. For the two-user case, we adopt the same rank-deficient channel model as in [13] in which the number of transmit and receive antennas and the ranks of channel matrices are arbitrary. We develop an achievable scheme and also derive a matching upper bound, thus characterizing the sum DoF. For the three-user case, we focus on a symmetric case in which each node has the same number of antennas, the rank of each direct link is the same, and the rank of each cross link is the same. We establish the achievable sum DoF of this channel by developing a new achievable scheme. The proposed scheme employs interference alignment to efficiently utilize the dimension of the received signal space when the rank of cross links is sufficiently large as compared to the number of antennas at each node. Furthermore, we derive an upper bound for the general K-user case, which is indeed achievable when the number of antennas at each

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[1]However, recently it has been shown in [7] that for *multihop* networks, feedback can increase DoF.

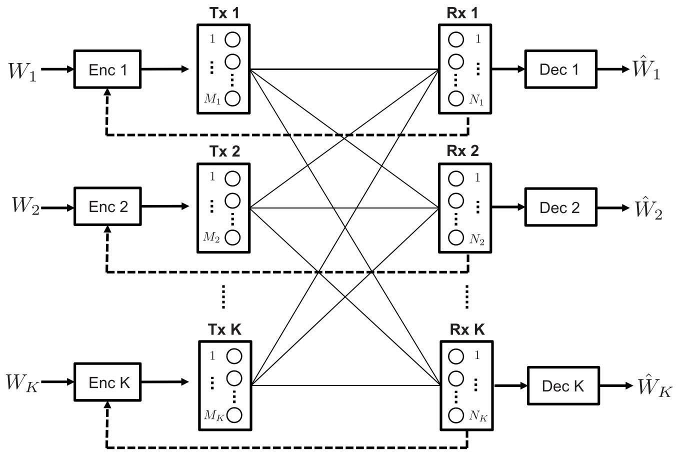

Fig. 1. The K-user rank-deficient interference channel with feedback.

node is sufficiently large. As a consequence of these results, we show that feedback can increase the DoF when channel matrices of desired links are highly rank-deficient. The gain comes from the fact that feedback can provide alternative signal paths in the rank-deficient channel, and hence the ranks of desired channel matrices are effectively increased, which is not possible in the full-rank channel. The result of this paper also includes that of the full-rank channel with feedback as a special case.

The rest of this paper is organized as follows. In Section II, we describe the channel model considered in the paper. In Section III, we show the main results of the paper and provide an intuition as to how feedback can increase the DoF in the rank-deficient channel. In addition, we provide the proofs of the main theorems in Sections IV, V, and VI. Finally, we conclude the paper in Section VII.

Notations: Throughout the paper, we will use $\mathbf{A}$ and $\mathbf{a}$ to denote a matrix and a vector, respectively. Let $\mathbf{A}^T$, $\mathbf{A}^*$, and $||\mathbf{A}||$ denote the transpose, the complex conjugate transpose, and the norm of $\mathbf{A}$, respectively. In addition, let $|\mathbf{A}|$ and $\text{rank}(\mathbf{A})$ denote the determinant and the rank of $\mathbf{A}$, respectively. The notations $\mathbf{I}_n$ and $\mathbf{0}_{n\times n}$ denote the $n \times n$ identity matrix and zero matrix, respectively. The notation $\mathbf{A} \preceq \mathbf{B}$ means that any linear combination of column vectors in $\mathbf{A}$ can be expressed by a linear combination of column vectors in $\mathbf{B}$. The notation $K_{\mathbf{x}}^G$ denotes the covariance matrix of a Gaussian random vector $\mathbf{x}$. Furthermore, we write $f(x) = o(x)$ if $\lim_{x\to\infty} \frac{f(x)}{x} = 0$. For convenience, when indexing channel matrices, we use modular arithmetic where the modulus is the number of users. (e.g., for the three-user case, $\mathbf{H}_{1,4}$ means $\mathbf{H}_{1,1}$).

II. System Model

Consider the K-user rank-deficient interference channel with feedback, as depicted in Fig. 1. Transmitter i wishes to communicate with receiver i, and transmitter i and receiver i use M_i and N_i antennas, respectively. We assume that all channel coefficients are fixed and known to all nodes. Then, the input and output relationship at time slot t is given by

$$\mathbf{y}_j(t) = \sum_{i=1}^{K} \mathbf{H}_{j,i}\mathbf{x}_i(t) + \mathbf{z}_j(t),$$

where $\mathbf{x}_i(t)$ is the $M_i \times 1$ input signal vector at transmitter i, $\mathbf{H}_{j,i}$ is the $N_j \times M_i$ channel matrix from transmitter i to receiver j, and $\mathbf{y}_j(t)$ is the $N_j \times 1$ received signal vector at receiver j. The noise vector $\mathbf{z}_j(t)$ is the additive white circularly symmetric complex Gaussian with zero mean and covariance of $\mathbf{I}_{N_j}$. We assume that all of the noise vectors and signal vectors are independent of each other.

In this paper, we adopt the rank-deficient channel model in [14], in which there are $D_{j,i} \leq \min\{M_i, N_j\}$ independent signal paths from transmitter i to receiver j. Let $\mathbf{H}_{j,i}^{(k)}$ denote the channel matrix corresponding to the kth signal path between transmitter i and receiver j. Note that due to the key-hole effect [14], $\text{rank}(\mathbf{H}_{j,i}^{(k)}) = 1$, $\forall k = 1, 2, \ldots, D_{j,i}$. Therefore, we assume that the matrix $\mathbf{H}_{j,i}$ is given by

$$\begin{aligned}\mathbf{H}_{j,i} &= \sum_{k=1}^{D_{j,i}} \mathbf{H}_{j,i}^{(k)} \\ &= \sum_{k=1}^{D_{j,i}} \mathbf{a}_{j,i}^{(k)} \mathbf{b}_{j,i}^{(k)T}, \quad \forall i, j = 1, 2, \ldots, K \end{aligned} \tag{1}$$

where $\mathbf{a}_{j,i}^{(k)}$ and $\mathbf{b}_{j,i}^{(k)}$ are $N_j \times 1$ and $M_i \times 1$ vectors respectively, and their coefficients are drawn from a continuous distribution. From (1), we can see that $\text{rank}(\mathbf{H}_{j,i}) = D_{j,i}$ with probability one.

There are K independent messages $W_1, W_2, \ldots, W_K$. At time slot t, transmitter i sends the encoded signal $\mathbf{x}_i(t)$, which is a function of W_i and past output sequences

$$\mathbf{y}_i^{t-1} \triangleq [\,\mathbf{y}_i^T(1) \quad \mathbf{y}_i^T(2) \quad \cdots \quad \mathbf{y}_i^T(t-1)]^T.$$

We assume that each transmitter should satisfy the average power constraint P, i.e., $E[|\mathbf{x}_i(t)|^2] \leq P$ for $i \in \{1, 2, \ldots, K\}$. A rate tuple $(R_1, R_2, \ldots, R_K)$ is said to be achievable if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, \ldots, , 2^{nR_K}, n)$ codes such that the average probability of decoding error tends to zero as the code length n goes to infinity. The capacity region $\mathcal{C}$ of this channel is the closure of the set of achievable rate tuples $(R_1, R_2, \ldots, R_K)$. The sum DoF is defined as $\Gamma = \lim_{P\to\infty} \max_{(R_1,R_2,\ldots,R_K)\in\mathcal{C}} \frac{\sum_{i=1}^{K} R_i}{\log(P)}$, which provides the sum capacity approximation at high SNR as

$$C_{\text{sum}}(P) = \max_{(R_1,R_2,\ldots,R_K)\in\mathcal{C}} \sum_{i=1}^{K} R_i = \Gamma \log(P) + o(\log(P)).$$

III. Main Results

A. Two-User Case

For the two-user case, we completely characterize the sum DoF as stated in the following theorem by developing an achievable scheme and deriving a matching upper bound.

Theorem 1 (Two-User Case): For the two-user rank-deficient interference channel with feedback, the sum DoF is given by

$$\begin{aligned}\Gamma_{fb} = \min\{&M_1 + N_2 - D_{2,1}, M_2 + N_1 - D_{1,2}, \\ &D_{1,1} + D_{2,2} + D_{1,2}, D_{1,1} + D_{2,2} + D_{2,1}, \\ &\min\{M_1, N_1\} + D_{2,2}, \min\{M_2, N_2\} + D_{1,1}\}\end{aligned}$$

for almost all values of channel coefficients.

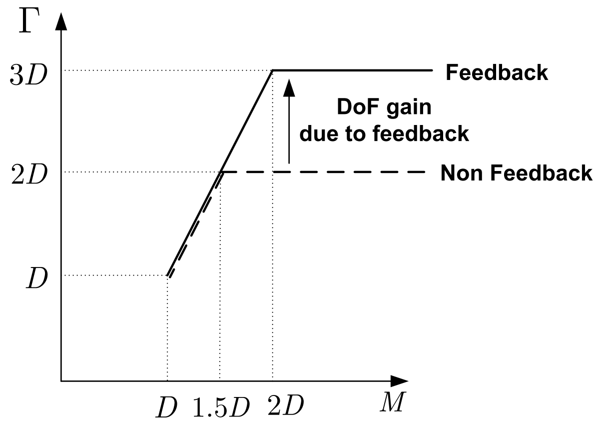

Fig. 2. Sum DoF for the two-user case when $M_1 = M_2 = N_1 = N_2 = M$ and $D_{1,1} = D_{1,2} = D_{2,1} = D_{2,2} = D$.

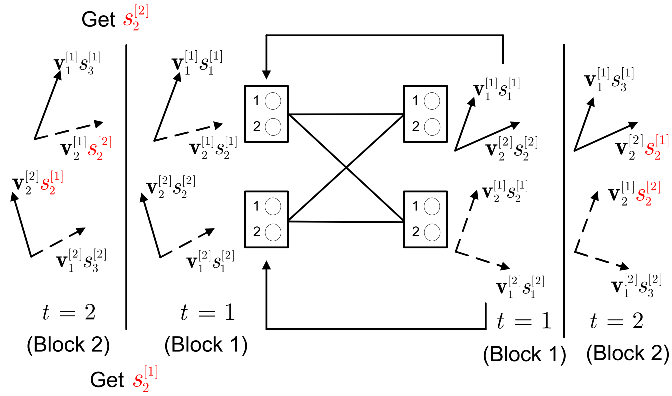

Fig. 3. Achievability in Example 1. The beamforming vectors represented by solid and dashed lines denote the signals transmitted to receivers 1 and 2, respectively.

Proof: See Section IV for the proof. ∎

Remark 1 (Full-Rank Case): For the case in which all the channel matrices have full rank, i.e., $D_{j,i} = \min(M_i, N_j)$ $\forall i, j = 1, 2$, the sum DoF becomes

$$\Gamma_{fb} = \min\{M_1 + M_2, N_1 + N_2, \max\{M_1, N_2\}, \max\{M_2, N_1\}\},$$

which coincides with the result for the full-rank interference channel in [5] and [6].

Remark 2: If all the direct links have full rank, i.e., $D_{1,1} = \min(M_1, N_1)$ and $D_{2,2} = \min(M_2, N_2)$, the result recovers the non-feedback case in [13]:

$$\Gamma_{no} = \min\{M_1 + N_2 - D_{2,1}, N_1 + M_2 - D_{1,2}, D_{1,1} + D_{2,2}\}.$$

Notice that for the above two cases, feedback cannot increase the sum DoF.

DoF Gain Due to Feedback: Consider a symmetric case where $M_1 = M_2 = N_1 = N_2 = M$ and $D_{1,1} = D_{1,2} = D_{2,1} = D_{2,2} = D$. We plot[2] the sum DoF as a function of M with fixed D in Fig. 2. Note that the DoF gain due to feedback can be achieved when the ratio of the number of antennas at each node to the rank of each channel matrix is greater than a certain threshold. For $M > 1.5D$, we can achieve a higher DoF. The gain comes from the fact that when the number of antennas at each node is large enough as compared to the channel ranks, feedback can provide alternative signal paths, thus increasing the ranks of effective desired channel matrices. In addition, we can see that the DoF gain saturates with respect to M. This is due to the fact that the ranks of effective desired channel matrices saturate when the rank of each matrix is fixed and M is greater than a certain threshold.

Here, we provide an intuition behind this gain through a simple example.

Example 1: Consider the case where $M_1 = M_2 = N_1 = N_2 = 2$ and $D_{1,1} = D_{2,2} = D_{1,2} = D_{2,1} = 1$. Our achievable scheme operates in two time slots.[3] See Fig. 3. Let $s_j^{[i]} \sim \mathcal{CN}(0, P/2)$ denote the jth symbol encoded for W_i, $i \in \{1, 2\}$. At time slot 1, we design the transmitted signals as

$$\mathbf{x}_1(1) = \mathbf{v}_1^{[1]} s_1^{[1]} + \mathbf{v}_2^{[1]} s_2^{[1]}$$
$$\mathbf{x}_2(1) = \mathbf{v}_1^{[2]} s_1^{[2]} + \mathbf{v}_2^{[2]} s_2^{[2]},$$

where the beamforming vectors are designed such that $\mathbf{H}_{2,1}\mathbf{v}_1^{[1]} = \mathbf{H}_{1,1}\mathbf{v}_2^{[1]} = \mathbf{H}_{2,2}\mathbf{v}_2^{[2]} = \mathbf{H}_{1,2}\mathbf{v}_1^{[2]} = 0$. Note that this design is feasible as the number of antennas at each node is greater than the rank of each channel matrix. Then, the received signal at each receiver is given by

$$\mathbf{y}_1(1) = \mathbf{H}_{1,1}\mathbf{v}_1^{[1]} s_1^{[1]} + \mathbf{H}_{1,2}\mathbf{v}_2^{[2]} s_2^{[2]} + \mathbf{z}_1(1)$$
$$\mathbf{y}_2(1) = \mathbf{H}_{2,2}\mathbf{v}_1^{[2]} s_1^{[2]} + \mathbf{H}_{2,1}\mathbf{v}_2^{[1]} s_2^{[1]} + \mathbf{z}_2(1).$$

Here, we have

$$\text{rank}\left(\begin{bmatrix} \mathbf{H}_{1,1}\mathbf{v}_1^{[1]} & \mathbf{H}_{1,2}\mathbf{v}_2^{[2]} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} \mathbf{H}_{2,2}\mathbf{v}_1^{[2]} & \mathbf{H}_{2,1}\mathbf{v}_2^{[1]} \end{bmatrix}\right) = 2$$

Therefore, receiver 1 can get $s_1^{[1]}$ and transmitter 1 can get $s_2^{[2]}$ after receiving $\mathbf{y}_1(1)$. Similarly, receiver 2 and transmitter 2 can get $s_1^{[2]}$ and $s_2^{[1]}$, respectively.

Now the idea is that at the next time slot, each transmitter sends the other user's symbol in addition to its own fresh symbol. To achieve this, we design the transmitted signals at time slot 2 as

$$\mathbf{x}_1(2) = \mathbf{v}_1^{[1]} s_3^{[1]} + \mathbf{v}_2^{[1]} s_2^{[2]}$$
$$\mathbf{x}_2(2) = \mathbf{v}_1^{[2]} s_3^{[2]} + \mathbf{v}_2^{[2]} s_2^{[1]},$$

where $s_3^{[1]}$ and $s_3^{[2]}$ are new symbols for users 1 and 2, respectively. Then we can see that receiver 1 can get $(s_2^{[1]}, s_3^{[1]})$ and receiver 2 can get $(s_2^{[2]}, s_3^{[2]})$. As a result, six symbols can be transmitted over two time slots, thus achieving $\Gamma_{fb} \geq 3$. This shows an improvement over the non-feedback DoF of 2.

Remark 3: From the above example, we see that feedback can create new signal paths (e.g., for $s_2^{[1]}$, transmitter 1 → receiver 2 → feedback → transmitter 2 → receiver 1), which do not exist in the non-feedback case. When the number of antennas at each node is large enough as compared to the ranks of channel matrices, the dimension of signal space at each

[2]Here, we assume that D is even, but we can get a similar graph when D is odd.

[3]In the achievability proofs of this paper, the time slot actually indicates a block of signals, but for notational simplicity the block notation is not used.

node becomes sufficiently large such that some signals can be transmitted through these new signal paths, thus increasing the ranks of effective desired channel matrices. For instance, the effective desired channel matrix for user 1 at time slot 2 is given by

$$\mathbf{H}^e_{1,1} = \begin{bmatrix} \mathbf{H}_{1,1} \\ \mathbf{H}_{1,2} \end{bmatrix},$$

where $\text{rank}(\mathbf{H}^e_{1,1}) = 2$. However, when all the direct links have full rank, feedback cannot increase the sum DoF since we cannot increase the ranks of direct links further and cannot create such alternative signal paths. In addition, we can see that if the rank of each channel matrix is fixed, having more antennas cannot further increase the DoF. This is because the rank of $\mathbf{H}^e_{1,1}$ remains the same when M is greater than a certain threshold. For example, for fixed $D_{1,1} = D_{1,2} = D$, $\text{rank}(\mathbf{H}^e_{1,1}) = 2D$ for all $M \geq 2D$. Note that the role of feedback here is similar to that of relays in [15], which shows that using multiple relays can create alternative signal paths, thus increasing the sum DoF in the rank-deficient interference channel.

B. Three-User Case

When $K \geq 3$, we focus on a *symmetric* case where $M_i = N_i = M$, $D_{i,i} = D_d$, and $D_{j,i} = D_c$, $\forall i = 1, 2, \ldots, K$ and $i \neq j$. Specifically, for $K = 3$, we develop a new achievable scheme which employs interference alignment when the rank of cross links D_c is sufficiently large. The achievable sum DoF for the three-user case is stated in the following theorem.

Theorem 2 (Lower Bound for $K = 3$): For the symmetric three-user rank-deficient interference channel with feedback, the following sum DoF is achievable for almost all values of channel coefficients.

$$\Gamma_{fb} \geq \begin{cases} M + D_d & \text{if } 2D_d \leq M \leq \min\{2D_c, D_d + D_c\}, \\ 2M - D_c & \text{if } \max\{2D_d, D_d + D_c\} \leq M \leq 2D_c, \\ \dfrac{3M}{2} & \text{if } M \leq \min\{2D_c, 2D_d\}, \\ 3M - 3D_c & \text{if } 2D_c \leq M \leq 2D_c + D_d, \\ 3D_d + 3D_c & \text{if } 2D_c + D_d \leq M. \end{cases}$$

Proof: See Section V for the proof. ■

Remark 4: If all the direct links have full rank, i.e., $D_d = M$, the result again recovers the non-feedback case in [13]:

$$\Gamma_{no} \geq \begin{cases} \dfrac{3M}{2} & \text{if } D_c \leq M \leq 2D_c, \\ 3M - 3D_c & \text{if } 2D_c \leq M. \end{cases}$$

Remark 5: As will be explained in Section V, our achievable scheme involves interference alignment with feedback when $D_c \leq M < 2D_c$, while it is merely based on zero-forcing when $M \geq 2D_c$. This is due to the fact that when the ratio of D_c to M is greater than a certain threshold, we cannot null out all the interference signals, and thus aligning unwanted signals is required to utilize the dimension of the received signal space more efficiently. Furthermore, as will be shown in Theorem 3, the proposed scheme achieves the optimal sum DoF when $M \geq 2D_c + D_d$.

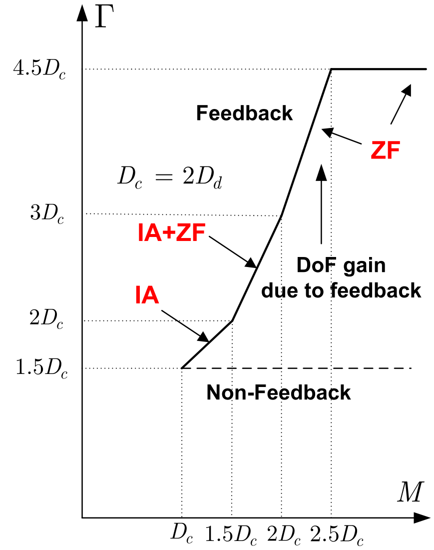

Fig. 4. Achievable sum DoF for the three-user case when $D_c = 2D_d$. The achievable scheme is based on zero-forcing (ZF) and/or interference alignment (IA), depending on the number of antennas at each node.

DoF Gain Due to Feedback: Consider the case where $D_c = 2D_d$. Again, we plot the sum DoF as a function of M with fixed D_c and D_d in Fig. 4. Note that for the three-user case, we employ interference alignment when the rank of each cross link D_c is sufficiently large as compared to the number of antennas at each node M (Here, when $M \leq 2D_c$). In addition, we can see that the slope in Fig. 4 increases with the number of antennas. This is because if there are enough antennas at each node, we can even create new interference-free signal paths via zero-forcing rather than aligning unwanted signals. Furthermore, as in the two-user case, the DoF gain saturates with respect to M due to the fact that the ranks of effective desired channel matrices saturate.

We provide a simple example that shows how interference alignment can be applied with feedback.

Example 2: Consider the case where $M = D_c = 5$ and $D_d = 1$. As in the two-user case, the proposed scheme operates in two time slots. See Fig. 5. At time slot 1, we design the transmitted signal for transmitter $i \in \{1, 2, 3\}$ as

$$\mathbf{x}_i(1) = \mathbf{v}_1^{[i]} s_1^{[i]} + \mathbf{v}_2^{[i]} s_2^{[i]} + \mathbf{v}_3^{[i]} s_3^{[i]}.$$

Here, transmitter i delivers $s_1^{[i]}$, $s_2^{[i]}$, and $s_3^{[i]}$ to receivers i, $i+1$, and $i+2$, respectively, while aligning unwanted signals for each receiver. Note that although $s_2^{[i]}$ and $s_3^{[i]}$ are not intended symbols for receivers $i+1$ and $i+2$, using feedback, transmitters $i+1$ and $i+2$ will forward them to receiver i in the next time slot.

To achieve these, we construct $\mathbf{v}_1^{[i]}$ such that

$$\mathbf{H}_{i+1,i}\mathbf{v}_1^{[i]} \doteq \mathbf{H}_{i+1,i+2}\mathbf{v}_1^{[i+2]},$$

where $\mathbf{a} \doteq \mathbf{b}$ means that $\mathbf{a}$ and $\mathbf{b}$ are collinear. We also design $\mathbf{v}_2^{[i]}$ and $\mathbf{v}_3^{[i]}$ such that

$$\mathbf{H}_{i,i}\mathbf{v}_3^{[i]} = \mathbf{H}_{i,i}\mathbf{v}_2^{[i]} = 0,$$
$$\mathbf{H}_{i+1,i}\mathbf{v}_3^{[i]} = \mathbf{H}_{i+1,i+2}\mathbf{v}_2^{[i+2]}.$$

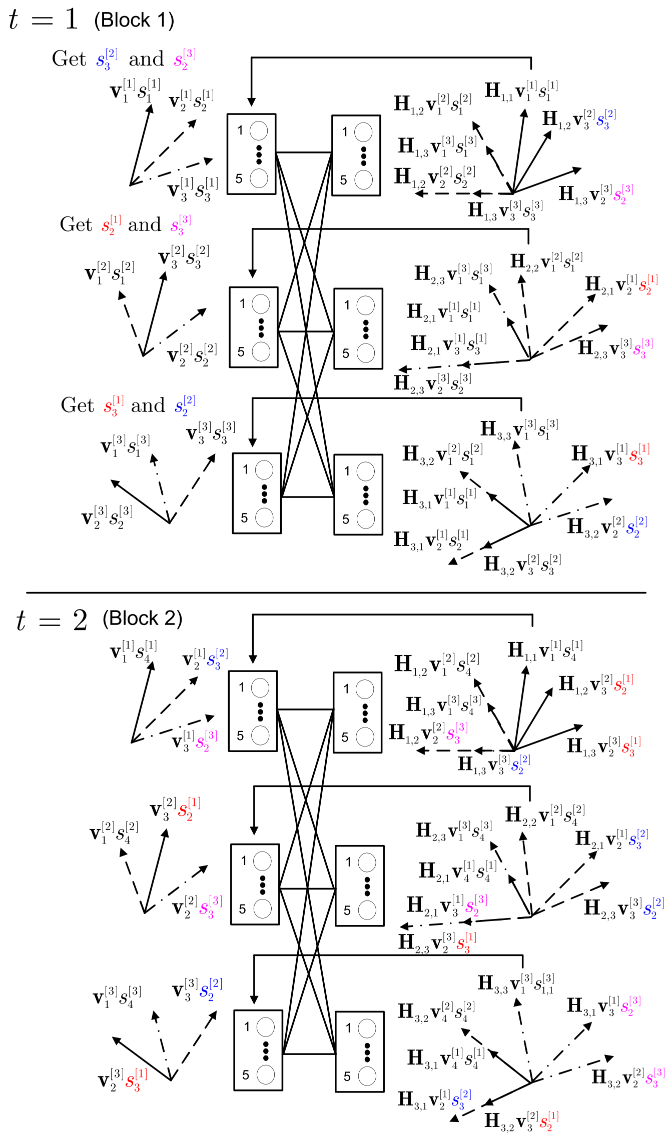

Fig. 5. Achievability in Example 2. The beamforming vectors represented by solid, dashed, and dashed-dotted lines denote the desired signals for receivers 1, 2, and 3 respectively at each time slot. Note that arrows in the figure represent linearly independent directions in a five dimensional space.

This beamforming design is feasible for $M \geq 2D_d$ and $M \leq 2D_c$ (This will be clarified in Section V.). It turns out that for receiver $i+1$, unwanted symbols $(s_1^{[i]}, s_1^{[i+2]})$ and $(s_3^{[i]}, s_2^{[i+2]})$ are aligned. Now we can see that

$$\text{rank}\left(\begin{bmatrix} \mathbf{H}_{i,i+1}\mathbf{v}_1^{[i+1]} & \mathbf{H}_{i,i+2}\mathbf{v}_1^{[i+2]} \end{bmatrix}\right) = 1,$$
$$\text{rank}\left(\begin{bmatrix} \mathbf{H}_{i,i+1}\mathbf{v}_2^{[i+1]} & \mathbf{H}_{i,i+2}\mathbf{v}_3^{[i+2]} \end{bmatrix}\right) = 1.$$

Hence, receiver i can get $s_1^{[i]}$. (See Fig. 5.)

On the other hand, transmitter i can get $s_2^{[i+2]}$ and $s_3^{[i+1]}$ after receiving $\mathbf{y}_i(1)$. At the next time slot, each transmitter forwards the other user's symbols in addition to its own fresh symbol. To achieve this, we design the transmitted signal as

$$\mathbf{x}_i(2) = \mathbf{v}_1^{[i]} s_4^{[i]} + \mathbf{v}_2^{[i]} s_3^{[i+1]} + \mathbf{v}_3^{[i]} s_2^{[i+2]},$$

where $s_4^{[i]}$ is a new symbol for user i. Then, using the same argument above, we can see that receiver i can get $(s_2^{[i]}, s_3^{[i]}, s_4^{[i]})$, $\forall i = 1, 2, 3$. As a result, we can send 12 symbols over two time slots, thus achieving $\Gamma_{fb} \geq 6$. Note that the sum DoF becomes three when there is no feedback.

C. Upper Bound for the K-User Case

Theorem 3 (Upper Bound for the K-User Case): For the symmetric K-user rank-deficient interference channel with feedback, the sum DoF is upper bounded by

$$\Gamma_{fb} \leq KD_d + \frac{D_c K(K-1)}{2}.$$

Proof: See Section VI for the proof. ■

Corollary 1: For the symmetric K-user rank-deficient interference channel with feedback, the sum DoF is given by

$$\Gamma_{fb} = KD_d + \frac{D_c K(K-1)}{2}$$

when $M \geq D_d + (K-1)D_c$.

Proof: The converse follows from Theorem 3. For achievability, we consider a simple extension of the scheme in Theorem 2. At the first time slot, each transmitter totally sends $D_d + (K-1)D_c$ symbols, in which D_d symbols are sent through the direct link and D_c symbols are sent through each cross link. Then, after receiving $\mathbf{y}_i(1)$, transmitter i and receiver i can get D_d of desired symbols and $(K-1)D_c$ of the other user's symbols, respectively. This is possible due to the fact that $M \geq D_d + (K-1)D_c$. At the second time slot, each transmitter sends its new D_d symbols and also forwards the other user's symbols to the corresponding receivers. Consequently, we can see that each receiver can get $2D_d + (K-1)D_c$ symbols during two-time slots, thus achieving $\Gamma_{fb} \geq KD_d + \frac{D_c K(K-1)}{2}$. ■

Remark 6: Suppose there are sufficiently many antennas at each node (e.g., $M \geq D_d + (K-1)D_c$). Then, from Corollary 1, we can see that the DoF gain due to feedback increases with the number of users. Let Γ_{fb} and Γ_{no} denote the sum DoFs when there is feedback and no feedback, respectively. In addition, consider the case where $D_c = D_d = D$. Then, we have

$$\frac{\Gamma_{fb}}{\Gamma_{no}} = \frac{DK(K+1)/2}{DK} = \frac{K+1}{2}$$

and we can see that the DoF gain is proportional to the number of users.

IV. Proof of Theorem 1

A. Achievability

For brevity, we first categorize beamforming vectors for transmitter $i \in \{1, 2\}$ into three types:

$$\mathbf{V}^{[i]} = \begin{bmatrix} \mathbf{V}_1^{[i]} & \mathbf{V}_2^{[i]} & \mathbf{V}_3^{[i]} \end{bmatrix},$$

where $\mathbf{V}_j^{[i]}$ is a concatenation of type j beamforming vectors of transmitter i, i.e.,

$$\mathbf{V}_j^{[i]} = \begin{bmatrix} \mathbf{v}_{j,j_1}^{[i]} & \mathbf{v}_{j,j_2}^{[i]} & \cdots & \mathbf{v}_{j,j_{d_j^{[i]}}}^{[i]} \end{bmatrix}, \quad \forall j = 1, 2, 3,$$

$d_j^{[i]}$ denotes the number of vectors in $\mathbf{V}_j^{[i]}$, $d_0^{[i]} = 0$, and $j_l = \sum_{k=1}^{j} d_{k-1}^{[i]} + l$.

- $\mathbf{v}_{1,k}^{[i]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $\mathbf{H}_{j,i}$, i.e., $\mathbf{H}_{j,i}\mathbf{v}_{1,k}^{[i]} = 0$, and $\mathbf{H}_{i,i}\mathbf{v}_{1,k}^{[i]} \neq 0$, where $i \neq j$. Note that since rank$(\mathbf{H}_{j,i}) = D_{j,i}$, the maximum number of beamforming vectors satisfying this condition is $M_i - D_{j,i}$.
- $\mathbf{v}_{2,k}^{[i]}$ denotes the kth beamforming vector for transmitter i whose coefficients are randomly generated from a continuous distribution and $0 < ||\mathbf{v}_{2,k}^{[i]}|| \leq \alpha$, where α is a finite value. Hence, $\mathbf{H}_{i,i}\mathbf{v}_{2,k}^{[i]} \neq 0$ and $\mathbf{H}_{j,i}\mathbf{v}_{2,k}^{[i]} \neq 0$ with probability one.
- $\mathbf{v}_{3,k}^{[i]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $\mathbf{H}_{i,i}$, i.e., $\mathbf{H}_{i,i}\mathbf{v}_{3,k}^{[i]} = 0$, and $\mathbf{H}_{j,i}\mathbf{v}_{3,k}^{[i]} \neq 0$. Note that since rank$(\mathbf{H}_{i,i}) = D_{i,i}$, the maximum number of beamforming vectors satisfying this condition is $M_i - D_{i,i}$.

Now we explain the proposed scheme. As shown in Example 1, our achievable scheme operates in two time slots. Let $s_j^{[i]} \sim \mathcal{CN}(0, P/d^{[i]})$ denote the jth symbol encoded for W_i, where $d^{[i]} = d_1^{[i]} + d_1^{[i]} + d_3^{[i]}$ is the number of transmitted symbols from transmitter i at each time slot. Then, the first time slot, we design the transmitted signal as

$$\begin{aligned}\mathbf{x}_i(1) &= \mathbf{V}^{[i]}\mathbf{s}^{[i]}(1)\\ &= \mathbf{V}_1^{[i]}\mathbf{s}_1^{[i]}(1) + \mathbf{V}_2^{[i]}\mathbf{s}_2^{[i]}(1) + \mathbf{V}_3^{[i]}\mathbf{s}_3^{[i]}(1),\end{aligned}$$

where

$$\mathbf{s}^{[i]}(1) = \begin{bmatrix}\mathbf{s}_1^{[i]}(1)\\ \mathbf{s}_2^{[i]}(1)\\ \mathbf{s}_3^{[i]}(1)\end{bmatrix} \sim \mathcal{CN}\left(\mathbf{0}_{d^{[i]}}, \frac{P}{d^{[i]}}\mathbf{I}_{d^{[i]}}\right),$$

$$\mathbf{s}_j^{[i]}(1) = \begin{bmatrix} s_{j_1}^{[i]} & s_{j_2}^{[i]} & \cdots & s_{j_{d_j^{[i]}}}^{[i]}\end{bmatrix}^T, \quad \forall j = 1, 2, 3.$$

Here, we set that $\mathbf{V}_1^{[i]}$, $\mathbf{V}_2^{[i]}$, and $\mathbf{V}_3^{[i]}$ are properly scaled to satisfy the power constraint P. Then the received signal at receiver $i \in \{1, 2\}$ is given by

$$\begin{aligned}\mathbf{y}_i(1) &= \mathbf{H}_{i,i}\mathbf{x}_i(1) + \mathbf{H}_{i,j}\mathbf{x}_j(1) + \mathbf{z}_i(1)\\ &= \mathbf{H}_{i,i}\mathbf{V}_1^{[i]}\mathbf{s}_1^{[i]}(1) + \mathbf{H}_{i,i}\mathbf{V}_2^{[i]}\mathbf{s}_2^{[i]}(1)\\ &\quad + \mathbf{H}_{i,j}\mathbf{V}_2^{[j]}\mathbf{s}_2^{[j]}(1) + \mathbf{H}_{i,j}\mathbf{V}_3^{[j]}\mathbf{s}_3^{[j]}(1) + \mathbf{z}_i(1)\end{aligned}$$

In the proposed scheme, we want to enable receiver i to obtain its desired symbols $\mathbf{s}_1^{[i]}(1)$ and $\mathbf{s}_2^{[i]}(1)$. In addition, we also want to make transmitter i be able to obtain the other user's symbols $\mathbf{s}_2^{[j]}(1)$ and $\mathbf{s}_3^{[j]}(1)$ after its corresponding receiver feeds back the received signal. To achieve these, we choose $d_1^{[1]}, d_2^{[1]}, d_3^{[1]}, d_1^{[2]}, d_2^{[2]}$, and $d_3^{[2]}$ to satisfy the following conditions.

$$d_3^{[1]} = d_3^{[2]} \triangleq d^f \tag{2}$$

$$0 \leq d_1^{[1]} \leq M_1 - D_{2,1} \tag{3}$$

$$0 \leq d_1^{[2]} \leq M_2 - D_{1,2} \tag{4}$$

$$0 \leq d_f \leq \min\{M_1 - D_{1,1}, M_2 - D_{2,2}\} \tag{5}$$

$$0 \leq d_1^{[1]} + d_2^{[1]} \leq D_{1,1} \tag{6}$$

$$0 \leq d_1^{[2]} + d_2^{[2]} \leq D_{2,2} \tag{7}$$

$$0 \leq d_2^{[1]} + d^f \leq D_{2,1} \tag{8}$$

$$0 \leq d_2^{[2]} + d^f \leq D_{1,2} \tag{9}$$

$$0 \leq d_2^{[1]} + d^f + d_1^{[2]} + d_2^{[2]} \leq N_2 \tag{10}$$

$$0 \leq d_2^{[2]} + d^f + d_1^{[1]} + d_2^{[1]} \leq N_1 \tag{11}$$

Here, the conditions (3)-(5) are due to the properties of $\mathbf{V}_1^{[i]}$ and $\mathbf{V}_3^{[i]}$; (6)-(9) are due to the fact that the number of symbols transmitted through a channel is constrained by the rank of the channel matrix; (10)-(11) are due to the fact that we have

$$\begin{aligned}&\text{rank}\left(\begin{bmatrix}\mathbf{H}_{i,i}\mathbf{V}_1^{[i]} & \mathbf{H}_{i,i}\mathbf{V}_2^{[i]} & \mathbf{H}_{i,j}\mathbf{V}_2^{[j]} & \mathbf{H}_{i,j}\mathbf{V}_3^{[j]}\end{bmatrix}\right)\\ &\qquad = \min\left\{d_1^{[i]} + d_2^{[i]} + d_2^{[j]} + d^f, N_i\right\}.\end{aligned}$$

for almost all values of channel coefficients, $\forall i = 1, 2$ and $i \neq j$. This is due to the facts that $\mathbf{V}^{[1]}$ and $\mathbf{V}^{[2]}$ are full-rank matrices and channel matrices are generic so that $\mathbf{H}_{i,i}$ and $\mathbf{H}_{i,j}$ are random linear transformations. In addition, since $d_1^{[i]}+d_2^{[i]} \leq D_{i,i}$ and $d_2^{[j]}+d^f \leq \min\{D_{1,2}, D_{2,1}\}$, linear independence of signals is also preserved. Thus, by observing $\mathbf{y}_i(1)$, receiver i and transmitter i can obtain the desired results, that is, receiver i and transmitter i can get $(\mathbf{s}_1^{[i]}(1), \mathbf{s}_2^{[i]}(1))$ and $(\mathbf{s}_2^{[j]}(1), \mathbf{s}_3^{[j]}(1))$ symbols, respectively, $\forall i = 1, 2$ and $i \neq j$.

Now we consider the proposed scheme in the second time slot. Recall that transmitter i can get the other user's symbols $\mathbf{s}_2^{[j]}(1)$ and $\mathbf{s}_3^{[j]}(1)$ after receiving feedback signal $\mathbf{y}_i(1)$. Among these signals, transmitter i tries to send only $\mathbf{s}_3^{[j]}(1)$ for receiver j at the next time slot since $\mathbf{s}_2^{[j]}(1)$ were already received by receiver j at the first time slot. Hence, at the second time slot, we design the transmitted signal as

$$\mathbf{x}_i(2) = \mathbf{V}_1^{[i]}\mathbf{s}_1^{[i]}(2) + \mathbf{V}_2^{[i]}\mathbf{s}_2^{[i]}(2) + \mathbf{V}_3^{[i]}\mathbf{s}_3^{[j]}(1),$$

where

$$\mathbf{s}_1^{[i]}(2) = \begin{bmatrix} s_{d_1^{[i]}+d_2^{[i]}+d_3^{[i]}+1}^{[i]} & \cdots & s_{2d_1^{[i]}+d_2^{[i]}+d_3^{[i]}}^{[i]}\end{bmatrix}^T,$$

$$\mathbf{s}_2^{[i]}(2) = \begin{bmatrix} s_{2d_1^{[i]}+d_2^{[i]}+d_3^{[i]}+1}^{[i]} & \cdots & s_{2d_1^{[i]}+2d_2^{[i]}+d_3^{[i]}}^{[i]}\end{bmatrix}^T.$$

Here, $\mathbf{s}_1^{[i]}(2)$ and $\mathbf{s}_2^{[i]}(2)$ are new symbols of user i transmitted at the second time slot. As a result, the received signal at receiver $i \in \{1, 2\}$ is given by

$$\begin{aligned}\mathbf{y}_i(2) &= \mathbf{H}_{i,i}\mathbf{x}_i(2) + \mathbf{H}_{i,j}\mathbf{x}_j(2) + \mathbf{z}_i(2)\\ &= \mathbf{H}_{i,i}\mathbf{V}_1^{[i]}\mathbf{s}_1^{[i]}(2) + \mathbf{H}_{i,i}\mathbf{V}_2^{[i]}\mathbf{s}_2^{[i]}(2)\\ &\quad + \mathbf{H}_{i,j}\mathbf{V}_2^{[j]}\mathbf{s}_2^{[j]}(2) + \mathbf{H}_{i,j}\mathbf{V}_3^{[j]}\mathbf{s}_3^{[i]}(1) + \mathbf{z}_i(2).\end{aligned}$$

Then, using the same argument as above, receiver i can obtain $\mathbf{s}_1^{[i]}(2)$, $\mathbf{s}_2^{[i]}(2)$, and $\mathbf{s}_3^{[i]}(1)$. As a result, we can see that transmitter i send $d_t^{[i]} = 2d_1^{[i]} + 2d_2^{[i]} + d_3^{[i]}$ symbols over two time slot. Therefore, the achievable sum DoF is

$$\begin{aligned}\Gamma_{fb} \geq \frac{d_t^{[1]} + d_t^{[2]}}{2} &= d_1^{[1]} + d_2^{[1]} + d_1^{[2]} + d_2^{[2]} + \frac{d_3^{[1]} + d_3^{[2]}}{2}\\ &= d_1^{[1]} + d_2^{[1]} + d_1^{[2]} + d_2^{[2]} + d_f.\end{aligned}$$

Finally, by evaluating the conditions (2)-(11) using the Fourier-Motzkin elimination, we get the desired bound:

$$\Gamma_{fb} \geq \min\{M_1 + N_2 - D_{2,1}, M_2 + N_1 - D_{1,2}, D_{1,1} + D_{2,2} + D_{1,2}, D_{1,1} + D_{2,2} + D_{2,1}, \min\{M_1, N_1\} + D_{2,2}, \min\{M_2, N_2\} + D_{1,1}\}.$$

Remark 7: The achievable sum DoF can also be established in an alternative way. One implicit strategy is to employ Lemma 1 in [4]. We can achieve the same DoF by setting $X_i = U_{if} + U_i + X_{ip}$ where $U_{if} = \mathbf{V}_3^{[i]}\mathbf{s}_3^{[i]}(1)$, $U = (U_{1f}, U_{2f})$, $U_i = \mathbf{V}_2^{[i]}\mathbf{s}_2^{[i]}(1)$, and $X_{ip} = \mathbf{V}_1^{[i]}\mathbf{s}_1^{[i]}(1)$, $\forall i = 1, 2$.

B. Converse

The proof is a direct extension of that in the two-user SISO interference channel with feedback [4]. Hence, we focus on explaining the steps needed for the rank-deficient channel.

Starting with Fano's inequality, we get:

$$\begin{aligned} n(R_1 + R_2 - \epsilon_n) &\leq I(W_1; \mathbf{y}_1^n) + I(W_2; \mathbf{y}_2^n) \\ &\leq I(W_1; \mathbf{y}_1^n, \mathbf{s}_1^n, W_2) + I(W_2; \mathbf{y}_2^n) \end{aligned}$$

where $\mathbf{s}_1 = \mathbf{H}_{2,1}\mathbf{x}_1 + \mathbf{z}_2$ as in [4]. Hence, by following the same steps in [4], we have

$$R_1 + R_2 \leq h(\mathbf{y}_2) + h(\mathbf{y}_1|\mathbf{s}_1, \mathbf{x}_2) - h(\mathbf{z}_1) - h(\mathbf{z}_2). \quad (12)$$

Now we evaluate the inequality (12) with respect to the number of antennas at each node and the rank of each channel matrix. From (12), we have

$$\begin{aligned} R_1 + R_2 &\leq h(\mathbf{y}_2) + h(\mathbf{y}_1|\mathbf{s}_1, \mathbf{x}_2) - h(\mathbf{z}_1) - h(\mathbf{z}_2) \\ &\leq h(\mathbf{y}_2) + h(\mathbf{H}_{1,1}\mathbf{x}_1 + \mathbf{z}_1|\mathbf{s}_1) - h(\mathbf{z}_1) - h(\mathbf{z}_2). \end{aligned}$$

Notice that

$$\mathbf{y}_2 = \begin{bmatrix} \mathbf{H}_{2,1} & \mathbf{H}_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \mathbf{z}_2.$$

Hence, we have

$$\begin{aligned} h(\mathbf{y}_2) - h(\mathbf{z}_2) - h(\mathbf{z}_1) &\overset{(a)}{\leq} \log\left[(\pi e)^{N_2}|K_{\mathbf{y}_2}^G|\right] - h(\mathbf{z}_2) - h(\mathbf{z}_1) \\ &\leq \log|K_{\mathbf{y}_2}^G| + o(\log(P)) \\ &\overset{(b)}{\leq} \min\{N_2, D_{2,2} + D_{2,1}\} + o(\log(P)) \end{aligned}$$

where (a) follows from the maximum differential entropy lemma for a complex random vector [3], [16], [17]; and (b) follows from the fact that $\text{rank}(\begin{bmatrix} \mathbf{H}_{2,1} & \mathbf{H}_{2,2} \end{bmatrix}) = \min\{N_2, D_{2,2} + D_{2,1}\}$.

Now we will derive an upper bound for $h(\mathbf{H}_{1,1}\mathbf{x}_1 + \mathbf{z}_1|\mathbf{s}_1)$. We have

$$\begin{aligned} h(\mathbf{H}_{1,1}\mathbf{x}_1 + \mathbf{z}_1|\mathbf{s}_1) &\overset{(a)}{\leq} \log|K_{\mathbf{H}_{1,1}\mathbf{x}_1+\mathbf{z}_1,\mathbf{s}_1}^G| - \log|K_{\mathbf{s}_1}^G| \\ &\overset{(b)}{=} \min\{D_{2,1} + D_{1,1}, M_1\}\log(P) \\ &\quad - D_{2,1}\log(P) + o(\log(P)) \\ &= \min\{D_{1,1}, M_1 - D_{2,1}\}\log(P) \\ &\quad + o(\log(P)), \end{aligned}$$

where (a) follows from a conditional version of the maximum differential entropy lemma [3], [16], [17]; and (b) follows from the fact that

$$\text{rank}\left(\begin{bmatrix} \mathbf{H}_{1,1} \\ \mathbf{H}_{2,1} \end{bmatrix}\right) = \min\{D_{2,1} + D_{1,1}, M_1\}.$$

Therefore, we have

$$\begin{aligned} \Gamma_{fb} &\leq \min\{N_2, D_{2,2} + D_{2,1}\} + \min\{M_1 - D_{2,1}, D_{1,1}\} \\ &= \min\{N_2 + M_1 - D_{2,1}, N_2 + D_{1,1}, \\ &\qquad M_1 + D_{2,2}, D_{2,2} + D_{2,1} + D_{1,1}\}. \end{aligned} \quad (13)$$

By symmetry, we can also get the following upper bound:

$$\Gamma_{fb} \leq \min\{N_1 + M_2 - D_{1,2}, N_1 + D_{2,2}, M_2 + D_{1,1}, D_{1,1} + D_{1,2} + D_{2,2}\}. \quad (14)$$

Combining (13) and (14), we get the desired bound:

$$\Gamma_{fb} \leq \min\{M_1 + N_2 - D_{2,1}, M_2 + N_1 - D_{1,2}, D_{1,1} + D_{2,2} + D_{1,2}, D_{1,1} + D_{2,2} + D_{2,1}, \min\{M_1, N_1\} + D_{2,2}, \min\{M_2, N_2\} + D_{1,1}\}.$$

V. Proof of Theorem 2

For this section, we categorize beamforming vectors for transmitter $i \in \{1, 2, 3\}$ into seven types:

$$\mathbf{V}^{[i]} = \begin{bmatrix} \mathbf{V}_1^{[i]} & \mathbf{V}_2^{[i]} & \cdots & \mathbf{V}_7^{[i]} \end{bmatrix}.$$

Here, since we consider the symmetric channel, we set $d_j^{[i]} = d_j$, $\forall i = 1, 2, 3$.

- $\mathbf{v}_{1,k}^{[i]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $\mathbf{H}_{i+1,i}$, i.e., $\mathbf{H}_{i+1,i}\mathbf{v}_{1,k}^{[i]} = 0$, and $\mathbf{H}_{i,i}\mathbf{v}_{1,k}^{[i]} \neq 0$ and $\mathbf{H}_{i+2,i}\mathbf{v}_{1,k}^{[i]} \neq 0$. Note that since $\text{rank}(\mathbf{H}_{i+1,i}) = D_c$, the maximum number of beamforming vectors satisfying this condition is $M - D_c$.
- $\mathbf{v}_{2,k}^{[i]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $\mathbf{H}_{i+2,i}$, i.e., $\mathbf{H}_{i+2,i}\mathbf{v}_{2,k}^{[i]} = 0$, and $\mathbf{H}_{i,i}\mathbf{v}_{2,k}^{[i]} \neq 0$ and $\mathbf{H}_{i+1,i}\mathbf{v}_{2,k}^{[i]} \neq 0$.
- $\mathbf{v}_{3,k}^{[i]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $[\,\mathbf{H}_{i+1,i}\ \mathbf{H}_{i+2,i}\,]$, i.e., $\mathbf{H}_{i+1,i}\mathbf{v}_{3,k}^{[i]} = 0$ and $\mathbf{H}_{i+2,i}\mathbf{v}_{3,k}^{[i]} = 0$, and $\mathbf{H}_{i,i}\mathbf{v}_{3,k}^{[i]} \neq 0$. Note that this type of beamforming vector exits only when $M \geq 2D_c$. Assuming $M \geq 2D_c$, the maximum number of beamforming vectors satisfying this condition is $M-2D_c$.
- After determining $\mathbf{V}_j^{[i]}$, $\forall j = 1, 2, 3$, we construct alignment beamforming vectors for each transmitter. We design $\mathbf{V}_4^{[i]}$ to satisfy

  $$\mathbf{H}_{i+1,i}\mathbf{V}_4^{[i]} \preceq \mathbf{H}_{i+1,i+2}\mathbf{V}_{i+2}^{I}$$

  where

  $$\mathbf{V}_i^I = \begin{bmatrix} \mathbf{V}_1^{[i]} & \mathbf{V}_4^{[i]} \end{bmatrix}.$$

  To construct feasible $\mathbf{V}_4^{[i]}$, we employ the beamforming scheme in [13], which is proposed for the non-feedback channel (Set $\mathbf{V}_4^{[i]} = V_i^A$ in [13]).

- Consider the case where $M \geq 2D_d$ and $M \leq 2D_c$. Let $\mathbf{V}_5^{[i]} = \left[\mathbf{V}_{5,1}^{[i]} \ \mathbf{V}_{5,2}^{[i]} \right]$, where

$$\mathbf{V}_{5,1}^{[i]} = \left[\mathbf{v}_{5,p+1}^{[i]} \quad \cdots \quad \mathbf{v}_{5,p+d_5/2}^{[i]} \right],$$
$$\mathbf{V}_{5,2}^{[i]} = \left[\mathbf{v}_{5,p+d_5/2+1}^{[i]} \quad \cdots \quad \mathbf{v}_{5,p+d_5}^{[i]} \right],$$

where $p = \sum_{j=1}^{4} d_j$. We construct alignment beamforming vectors in $\mathbf{V}_{5,1}^{[i]}$ and $\mathbf{V}_{5,2}^{[i]}$ such that

$$\mathbf{H}_{i,i}\mathbf{V}_{5,1}^{[i]} = \mathbf{H}_{i,i}\mathbf{V}_{5,2}^{[i]} = 0$$
$$\mathbf{H}_{i+2,i}\mathbf{V}_{5,1}^{[i]} = \mathbf{H}_{i+2,i+1}\mathbf{V}_{5,2}^{[i+1]},$$

or equivalently,

$$\underbrace{\begin{bmatrix} \mathbf{H}_{i+2,i} & -\mathbf{H}_{i+2,i+1} \\ \mathbf{H}_{i,i} & \mathbf{0}_{M\times M} \\ \mathbf{0}_{M\times M} & \mathbf{H}_{i+1,i+1} \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} \mathbf{V}_{5,1}^{[i]} \\ \mathbf{V}_{5,2}^{[i+1]} \end{bmatrix} = 0.$$

Since $\mathbf{T}$ is the $3M \times 2M$ matrix whose rank is $M + 2D_d$, we can find feasible $\mathbf{V}_{5,1}^{[i]}$ and $\mathbf{V}_{5,2}^{[i+1]}$, where $d_5 \leq 2M - 4D_d$. For the case where $M \leq 2D_d$ or $M \geq 2D_c$, we set $d_5 = 0$.
- $\mathbf{v}_{6,k}^{[i]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $[\,\mathbf{H}_{i,i}\ \mathbf{H}_{i+1,i}\,]$, i.e, $\mathbf{H}_{i,i}\mathbf{v}_{6,k}^{[i]} = 0$ and $\mathbf{H}_{i+1,i}\mathbf{v}_{6,k}^{[i]} = 0$, and $\mathbf{H}_{i+2,i}\mathbf{v}_{6,k}^{[i]} \neq 0$. Note that this type of beamforming vector exits only when $M \geq D_d + D_c$. Assuming $M \geq D_d + D_c$, the maximum number of beamforming vectors satisfying this condition is $M - D_d - D_c$.
- $\mathbf{v}_{7,k}^{[i]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $[\,\mathbf{H}_{i,i}\ \mathbf{H}_{i+2,i}\,]$, i.e, $\mathbf{H}_{i,i}\mathbf{v}_{7,k}^{[i]} = 0$ and $\mathbf{H}_{i+2,i}\mathbf{v}_{7,k}^{[i]} = 0$, and $\mathbf{H}_{i+1,i}\mathbf{v}_{7,k}^{[i]} \neq 0$.

Notice that $\mathbf{V}_4^{[i]}$ and $\mathbf{V}_5^{[i]}$ are alignment beamforming matrices while the others are zero-forcing beamforming matrices.

Now we explain the proposed scheme. At time slot t, we design the transmitted signal for transmitter $i \in \{1, 2, 3\}$ as

$$\mathbf{x}_i(t) = \sum_{j=1}^{7} \mathbf{V}_j^{[i]}\mathbf{s}_j^{[i]}(t),$$

where the beamforming vectors are properly scaled to satisfy the power constraint P and

$$\mathbf{s}^{[i]}(t) = \begin{bmatrix} \mathbf{s}_1^{[i]}(t) \\ \mathbf{s}_2^{[i]}(t) \\ \vdots \\ \mathbf{s}_7^{[i]}(t) \end{bmatrix} \sim \mathcal{CN}\left(\mathbf{0}_d, \frac{P}{d}\mathbf{I}_d\right),$$

where $d = \sum_{j=1}^{7} d_j$ is the number of transmitted symbols of each transmitter at each time slot. Then, due to the properties of $\mathbf{V}_j^{[i]}$, $j \in \{1, 2, \ldots, 7\}$, the received signal of receiver i at time slot 1 is given by

$$\begin{aligned}\mathbf{y}_i(1) &= \mathbf{H}_{i,i}\mathbf{x}_i(1) + \mathbf{H}_{i,i+1}\mathbf{x}_{i+1}(1) + \mathbf{H}_{i,i+2}\mathbf{x}_{i+2}(1) + \mathbf{z}_i(1) \\ &= \mathbf{H}_{i,i}\Big(\mathbf{V}_1^{[i]}\mathbf{s}_1^{[i]}(1) + \mathbf{V}_2^{[i]}\mathbf{s}_2^{[i]}(1) \\ &\qquad + \mathbf{V}_3^{[i]}\mathbf{s}_3^{[i]}(1) + \mathbf{V}_4^{[i]}\mathbf{s}_4^{[i]}(1)\Big) \\ &\quad + \mathbf{H}_{i,i+1}\Bigg(\mathbf{V}_1^{[i+1]}\mathbf{s}_1^{[i+1]}(1) + \mathbf{V}_4^{[i+1]}\mathbf{s}_4^{[i+1]}(1) \\ &\qquad + \mathbf{V}_5^{[i+1]}\begin{bmatrix} \mathbf{s}_{5,1}^{[i+1]}(1) \\ \mathbf{s}_{5,2}^{[i+1]}(1) \end{bmatrix} + \mathbf{V}_6^{[i+1]}\mathbf{s}_6^{[i+1]}(1)\Bigg) \\ &\quad + \mathbf{H}_{i,i+2}\Bigg(\mathbf{V}_2^{[i+2]}\mathbf{s}_2^{[i+2]}(1) + \mathbf{V}_4^{[i+2]}\mathbf{s}_4^{[i+2]}(1) \\ &\qquad + \mathbf{V}_5^{[i+2]}\begin{bmatrix} \mathbf{s}_{5,1}^{[i+2]}(1) \\ \mathbf{s}_{5,2}^{[i+2]}(1) \end{bmatrix} + \mathbf{V}_7^{[i+2]}\mathbf{s}_7^{[i+2]}(1)\Bigg) \\ &\quad + \mathbf{z}_i(1), \end{aligned} \tag{15}$$

where

$$\mathbf{s}_j^{[i]}(1) = \left[s_{j_1}^{[i]} \quad s_{j_2}^{[i]} \quad \cdots \quad s_{j_{d_j}}^{[i]} \right]^T, \quad \forall j = 1, 2, \ldots, 7,$$
$$\mathbf{s}_{5,1}^{[i]}(1) = \left[s_{p+1}^{[i]} \quad s_{p+2}^{[i]} \quad \cdots \quad s_{p+d_5/2}^{[i]} \right]^T,$$
$$\mathbf{s}_{5,2}^{[i]}(1) = \left[s_{p+d_5/2+1}^{[i]} \quad s_{p+d_5/2+2}^{[i]} \quad \cdots \quad s_{p+d_5}^{[i]} \right]^T,$$

$j_l = \sum_{k=1}^{j} d_{k-1} + l$, and $d_0 = 0$.

As in the two-user case, our achievable scheme operates in two time slots. However, unlike the two-user case, the achievable scheme employs interference alignment when D_c is sufficiently large. In the first time slot, transmitter i tries to deliver the symbols $(\mathbf{s}_1^{[i]}(1), \mathbf{s}_2^{[i]}(1), \mathbf{s}_3^{[i]}(1), \mathbf{s}_4^{[i]}(1))$, $(\mathbf{s}_{5,1}^{[i]}(1), \mathbf{s}_7^{[i]}(1))$, and $(\mathbf{s}_{5,2}^{[i]}(1), \mathbf{s}_6^{[i]}(1))$ to receivers i, $i+1$, and $i+2$, respectively. Here, although $(\mathbf{s}_{5,1}^{[i]}(1), \mathbf{s}_7^{[i]}(1))$ and $(\mathbf{s}_{5,2}^{[i]}(1), \mathbf{s}_6^{[i]}(1))$ are not intended symbols for receivers $i+1$ and $i+2$, using feedback, transmitters $i+1$ and $i+2$ will forward them to receiver i in the next time slot. To achieve this, we choose d_j to satisfy the following conditions:

$$0 \leq d_1 \leq M - D_c \tag{16}$$
$$0 \leq d_2 \leq M - D_c \tag{17}$$
$$0 \leq d_3 \leq \max(M - 2D_c, 0) \tag{18}$$
$$0 \leq d_5 \leq 2M - 4D_d \tag{19}$$
$$\text{for } M \geq 2D_d \text{ and } M \leq 2D_c \text{ (otherwise } d_5 = 0)$$
$$0 \leq d_6 \leq \max(M - D_c - D_d, 0) \tag{20}$$
$$0 \leq d_7 \leq \max(M - D_c - D_d, 0) \tag{21}$$
$$0 \leq d_1 + d_2 + d_3 + d_4 \leq D_d \tag{22}$$
$$0 \leq d_1 + d_4 + d_5 + d_6 \leq D_c \tag{23}$$
$$0 \leq d_2 + d_4 + d_5 + d_7 \leq D_c \tag{24}$$
$$0 \leq 2d_1 + 2d_2 + d_3 + 2d_4 + \frac{3}{2}d_5 + d_6 + d_7 \leq M \tag{25}$$
$$0 \leq \sum_{j=1}^{7} d_j \leq M \tag{26}$$

Here, the conditions (16)-(21) are due to the properties of $\mathbf{V}_j^{[i]}$; (22)-(24) are due to the fact that the number of symbols transmitted through a channel is constrained by the rank of the channel matrix; (25) is due to the fact that

$$\operatorname{rank}\left(\left[\begin{array}{lll}\mathbf{H}_{i,i+1}\mathbf{V}_1^{[i+1]} & \mathbf{H}_{i,i+1}\mathbf{V}_4^{[i+1]} & \mathbf{H}_{i,i+2}\mathbf{V}_4^{[i+2]}\end{array}\right]\right) = \min\{d_1+d_4, M\},$$
$$\operatorname{rank}\left(\left[\begin{array}{ll}\mathbf{H}_{i,i+1}\mathbf{V}_5^{[i+1]} & \mathbf{H}_{i,i+2}\mathbf{V}_5^{[i+2]}\end{array}\right]\right) = \min\left\{\frac{3}{2}d_5, M\right\},$$

because of the alignment properties of $\mathbf{V}_4^{[i]}$ and $\mathbf{V}_5^{[i]}$ so that

$$\operatorname{rank}\left(\left[\begin{array}{lll}\mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3\end{array}\right]\right) = \min\left\{2d_1+2d_2+d_3+2d_4+\frac{3}{2}d_5+d_6+d_7, M\right\}$$

where

$$\mathbf{A}_1 = \mathbf{H}_{i,i}\left[\begin{array}{llll}\mathbf{V}_1^{[i]} & \mathbf{V}_2^{[i]} & \mathbf{V}_3^{[i]} & \mathbf{V}_4^{[i]}\end{array}\right],$$
$$\mathbf{A}_2 = \mathbf{H}_{i,i+1}\left[\begin{array}{llll}\mathbf{V}_1^{[i+1]} & \mathbf{V}_4^{[i+1]} & \mathbf{V}_5^{[i+1]} & \mathbf{V}_6^{[i+1]}\end{array}\right],$$
$$\mathbf{A}_3 = \mathbf{H}_{i,i+2}\left[\begin{array}{llll}\mathbf{V}_2^{[i+2]} & \mathbf{V}_4^{[i+2]} & \mathbf{V}_5^{[i+2]} & \mathbf{V}_7^{[i+2]}\end{array}\right];$$

and (26) is due to the fact that the number of transmitted symbols from a transmitter should be less than or equal to the number of antennas at the transmitter.

Then, we have

$$\operatorname{rank}(\mathbf{A}_1) = \sum_{j=1}^{4} d_j, \tag{27}$$
$$\operatorname{rank}(\mathbf{A}_2) = d_1+d_4+d_5+d_6, \tag{28}$$
$$\operatorname{rank}(\mathbf{A}_3) = d_2+d_4+d_5+d_7, \tag{29}$$
$$\operatorname{rank}\left(\left[\begin{array}{ll}\mathbf{A}_2 & \mathbf{A}_3\end{array}\right]\right) = d_1+d_2+d_4+\frac{3}{2}d_5+d_6+d_7, \tag{30}$$
$$\operatorname{rank}\left(\left[\begin{array}{lll}\mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3\end{array}\right]\right) = 2d_1+2d_2+d_3+2d_4 + \frac{3}{2}d_5+d_6+d_7 \tag{31}$$

for almost all values of channel coefficients. Notice that (27)-(31) are due to the facts that $\mathbf{V}^{[1]}$, $\mathbf{V}^{[2]}$, and $\mathbf{V}^{[3]}$ are full-rank matrices and all the channel matrices are generic. Thus, by observing $\mathbf{y}_i(1)$, receiver i and transmitter i can get desired symbols $(\mathbf{s}_1^{[i]}(1), \mathbf{s}_2^{[i]}(1), \mathbf{s}_3^{[i]}(1), \mathbf{s}_4^{[i]}(1))$ and the other user's symbols $(\mathbf{s}_{5,2}^{[i+1]}(1), \mathbf{s}_6^{[i+1]}(1), \mathbf{s}_{5,1}^{[i+2]}(1), \mathbf{s}_7^{[i+2]}(1))$, respectively, as desired.

Now we consider the proposed scheme in the second time slot. Recall that transmitter i can obtain the other user's symbols $(\mathbf{s}_{5,2}^{[i+1]}(1), \mathbf{s}_6^{[i+1]}(1), \mathbf{s}_{5,1}^{[i+2]}(1), \mathbf{s}_7^{[i+2]}(1))$ after receiving $\mathbf{y}_i(1)$ via feedback. In the second time slot, each transmitter will forward these signals to the corresponding receivers, i.e., forward $(\mathbf{s}_{5,2}^{[i+1]}(1), \mathbf{s}_6^{[i+1]}(1))$ to receiver $i+1$ and $(\mathbf{s}_{5,1}^{[i+2]}(1), \mathbf{s}_7^{[i+2]}(1))$ to receiver $i+2$, and also send its own new symbols. To achieve this, we set the symbols of user i transmitted at time slot 2 as

$$\mathbf{s}_j^{[i]}(2) = \left[\begin{array}{lll} s_{j_1+d}^{[i]} & \cdots & s_{j_{d_j}+d}^{[i]}\end{array}\right]^T, \quad \forall j = 1,\ldots,4,$$
$$\mathbf{s}_5^{[i]}(2) = \begin{bmatrix}\mathbf{s}_{5,2}^{[i+1]}(1)\\ \mathbf{s}_{5,1}^{[i+2]}(1)\end{bmatrix},$$
$$\mathbf{s}_6^{[i]}(2) = \mathbf{s}_7^{[i+2]}(1),$$
$$\mathbf{s}_7^{[i]}(2) = \mathbf{s}_6^{[i+1]}(1).$$

Here, $\mathbf{s}_1^{[i]}(2)$, $\mathbf{s}_2^{[i]}(2)$, $\mathbf{s}_3^{[i]}(2)$, and $\mathbf{s}_4^{[i]}(2)$ are new symbols of user i transmitted at the second time slot. As a result, the received signal of receiver i at time slot 2 is given by

$$\begin{aligned}\mathbf{y}_i(2) &= \mathbf{H}_{i,i}\mathbf{x}_i(2)+\mathbf{H}_{i,i+1}\mathbf{x}_{i+1}(2)+\mathbf{H}_{i,i+2}\mathbf{x}_{i+2}(2)+\mathbf{z}_i(2)\\ &= \mathbf{H}_{i,i}\Big(\mathbf{V}_1^{[i]}\mathbf{s}_1^{[i]}(2)+\mathbf{V}_2^{[i]}\mathbf{s}_2^{[i]}(2)\\ &\qquad + \mathbf{V}_3^{[i]}\mathbf{s}_3^{[i]}(2)+\mathbf{V}_4^{[i]}\mathbf{s}_4^{[i]}(2)\Big)\\ &\quad + \mathbf{H}_{i,i+1}\Bigg(\mathbf{V}_1^{[i+1]}\mathbf{s}_1^{[i+1]}(2)+\mathbf{V}_4^{[i+1]}\mathbf{s}_4^{[i+1]}(2)\\ &\qquad + \mathbf{V}_5^{[i+1]}\begin{bmatrix}\mathbf{s}_{5,2}^{[i+2]}(1)\\ \mathbf{s}_{5,1}^{[i]}(1)\end{bmatrix}+\mathbf{V}_6^{[i+1]}\mathbf{s}_7^{[i]}(1)\Bigg)\\ &\quad + \mathbf{H}_{i,i+2}\Bigg(\mathbf{V}_2^{[i+2]}\mathbf{s}_2^{[i+2]}(2)+\mathbf{V}_4^{[i+2]}\mathbf{s}_4^{[i+2]}(2)\\ &\qquad + \mathbf{V}_5^{[i+2]}\begin{bmatrix}\mathbf{s}_{5,2}^{[i]}(1)\\ \mathbf{s}_{5,1}^{[i+1]}(1)\end{bmatrix}+\mathbf{V}_7^{[i+2]}\mathbf{s}_6^{[i]}(1)\Bigg)\\ &\quad + \mathbf{z}_i(2),\end{aligned}$$

Then, using the same argument as above, receiver i can obtain the symbols in $(\mathbf{s}_1^{[i]}(2), \mathbf{s}_2^{[i]}(2), \mathbf{s}_3^{[i]}(2), \mathbf{s}_4^{[i]}(2), \mathbf{s}_5^{[i]}(1), \mathbf{s}_6^{[i]}(1), \mathbf{s}_7^{[i]}(1))$.

Consequently, after two time slots, each transmitter can send $d_t = 2d_1+2d_2+2d_3+2d_4+d_5+d_6+d_7$ symbols, thus achieving the sum DoF:

$$\Gamma_{fb} \geq \frac{3d_t}{2} = 3\left(d_1+d_2+d_3+d_4+\frac{d_5+d_6+d_7}{2}\right).$$

Now we analyze the achievable sum DoF by determining suitable d_j, $\forall j = 1,2,\ldots,7$, with respect to M, D_d and D_c.

A. Case 1: When $D_c \leq M \leq 2D_c$

1) $M \geq 2D_d$ and $M \leq D_d+D_c$: In this case, the conditions (18), (20), (21), (22), and (25) respectively become

$$d_3 = 0,$$
$$d_6 = 0,$$
$$d_7 = 0,$$
$$0 \leq d_1+d_2+d_4 \leq D_d,$$
$$0 \leq 2d_1+2d_2+2d_4+\frac{3}{2}d_5 \leq M.$$

From above equalities and inequalities, we can obtain the following bound:

$$3\left(d_1+d_2+d_3+d_4+\frac{d_5+d_6+d_7}{2}\right) \leq M+D_d.$$

Notice that this bound is indeed achievable. To achieve this, we can set[4]

$$\begin{aligned} d_1 &= M - D_c, \\ d_4 &= D_d + D_c - M, \\ d_2 &= d_3 = d_6 = d_7 = 0, \\ d_5 &= \frac{2M - 4D_d}{3}, \end{aligned}$$

which also satisfy the conditions (16)-(26), thus achieving the following sum DoF:

$$\begin{aligned} \Gamma_{fb} &\geq 3\left(D_d + \frac{M - 2D_d}{3}\right) \\ &= M + D_d. \end{aligned} \tag{32}$$

Note that the proposed scheme involves interference alignment for this case ($d_4 \neq 0$ and $d_5 \neq 0$).

2) When $M \geq 2D_d$ and $M \geq D_d + D_c$: In this case, the conditions (18), (20), (21), (22), and (25) respectively become

$$\begin{aligned} & d_3 = 0, \\ & 0 \leq d_6 \leq M - D_c - D_d, \\ & 0 \leq d_7 \leq M - D_c - D_d, \\ & 0 \leq d_1 + d_2 + d_4 \leq D_d, \\ & 0 \leq 2d_1 + 2d_2 + 2d_4 + \frac{3}{2}d_5 + d_6 + d_7 \leq M. \end{aligned}$$

Thus, we can obtain the following bound:

$$3\left(d_1 + d_2 + d_3 + d_4 + \frac{d_5 + d_6 + d_7}{2}\right) \leq 2M - D_c.$$

To achieve this bound, we set

$$\begin{aligned} d_1 &= d_2 = \frac{D_d}{2}, \\ d_3 &= d_4 = 0, \\ d_5 &= \frac{4D_c - 2M}{3}, \\ d_6 &= d_7 = M - D_c - D_d, \end{aligned}$$

which also satisfy the conditions (16)-(26), thus achieving the sum DoF:

$$\begin{aligned} \Gamma_{fb} &\geq 3\left(M - D_c + \frac{2D_c - M}{3}\right) \\ &= 2M - D_c. \end{aligned} \tag{33}$$

As in the previous case, the proposed scheme involves interference alignment ($d_5 \neq 0$).

3) When $M \leq 2D_d$: In this case, the conditions (18), (19), (20), (21), and (25) respectively become

$$\begin{aligned} & d_3 = 0, \\ & d_5 = 0, \\ & d_6 = 0, \\ & d_7 = 0, \\ & 0 \leq 2d_1 + 2d_2 + 2d_4 \leq M. \end{aligned}$$

Hence, we can obtain the following bound:

$$3\left(d_1 + d_2 + d_3 + d_4 + \frac{d_5 + d_6 + d_7}{2}\right) \leq \frac{3M}{2}.$$

To achieve this bound, we employ the non-feedback scheme in [13] and can achieve

$$\Gamma_{fb} \geq \frac{3M}{2} \tag{34}$$

by setting

$$\begin{aligned} d_1 &= M - D_c, \\ d_4 &= D_c - \frac{M}{2}, \\ d_2 &= d_3 = d_5 = d_6 = d_7 = 0. \end{aligned}$$

Combining (32), (33), and (34), we obtain the following lower bound on the sum DoF.

$$\Gamma_{fb} \geq \begin{cases} M + D_d & \text{if } 2D_d \leq M \leq \min\{2D_c, D_d + D_c\}, \\ 2M - D_c & \text{if } \max\{2D_d, D_d + D_c\} \leq M \leq 2D_c \\ \frac{3M}{2} & \text{if } M \leq \min\{2D_c, 2D_d\}, \end{cases} \tag{35}$$

B. Case 2: When $2D_c \leq M \leq 2D_c + D_d$

1) When $M \geq D_c + D_d$: In this case, the conditions (18), (19), (20), (21), and (25) respectively become

$$\begin{aligned} & 0 \leq d_3 \leq M - 2D_c, \\ & d_5 = 0, \\ & 0 \leq d_6 \leq M - D_c - D_d, \\ & 0 \leq d_7 \leq M - D_c - D_d, \\ & 0 \leq 2d_1 + 2d_2 + d_3 + 2d_4 + d_6 + d_7 \leq M. \end{aligned}$$

Hence, we can obtain the following bound:

$$3\left(d_1 + d_2 + d_3 + d_4 + \frac{d_5 + d_6 + d_7}{2}\right) \leq 3M - 3D_c.$$

To achieve this bound, we set

$$\begin{aligned} d_1 &= d_2 = \frac{2D_c + D_d - M}{2}, \\ d_3 &= M - 2D_c, \\ d_4 &= d_5 = 0, \\ d_6 &= d_7 = M - D_c - D_d \end{aligned}$$

which satisfy the conditions (16)-(26). Then, the achievable sum DoF is given by

$$\begin{aligned} \Gamma_{fb} &\geq 3\left(D_d + M - D_c - D_d\right) \\ &= 3M - 3D_c. \end{aligned} \tag{36}$$

Note that the proposed scheme is merely based on zero forcing in this case ($d_4 = d_5 = 0$).

[4]If $\frac{2M-4D_d}{3}$ is not an integer, we consider the three-time symbol extension as in [13] and [18]. Furthermore, whenever d_j is not an integer, we can consider a proper symbol extension.

2) When $M \le D_d + D_c$: In this case, the conditions (18), (19), (20), (21), and (25) respectivley become

$$\begin{aligned} 0 &\le d_3 \le M - 2D_c, \\ d_5 &= 0, \\ d_6 &= 0, \\ d_7 &= 0, \\ 0 &\le 2d_1 + 2d_2 + d_3 + 2d_4 \le M. \end{aligned}$$

Therefore, we can obtain the following bound:

$$3\left(d_1 + d_2 + d_3 + d_4 + \frac{d_5 + d_6 + d_7}{2}\right) \le 3M - 3D_c$$

To achieve this bound, we employ the non-feedback scheme in [13] and can achieve

$$\Gamma_{fb} \ge 3M - 3D_c \tag{37}$$

by setting

$$\begin{aligned} d_1 &= d_2 = \frac{D_c}{2}, \\ d_3 &= M - 2D_c, \\ d_4 &= d_5 = d_6 = d_7 = 0. \end{aligned}$$

Combining (36) and (37), we obtain the following lower bound on the sum DoF.

$$\Gamma_{fb} \ge 3M - 3D_c, \quad \text{if } 2D_c \le M \le 2D_c + D_d. \tag{38}$$

C. Case 3: When $M \ge 2D_c + D_d$

In this case, we use only $2D_c + D_d$ antennas out of M antennas at each node. Then, from the result in Case 2, we can achieve

$$\Gamma_{fb} \ge 3D_d + 3D_c, \quad \text{if } M \ge 2D_c + D_d, \tag{39}$$

by setting

$$\begin{aligned} d_1 &= d_2 = d_4 = d_5 = 0, \\ d_3 &= D_d, \\ d_6 &= d_7 = D_c. \end{aligned}$$

Finally, by combining (35), (38), and (39), we get the desired bound:

$$\Gamma_{fb} \ge \begin{cases} M + D_d & \text{if } 2D_d \le M \le \min\{2D_c, D_d + D_c\}, \\ 2M - D_c & \text{if } \max\{2D_d, D_d + D_c\} \le M \le 2D_c, \\ \dfrac{3M}{2} & \text{if } M \le \min\{2D_c, 2D_d\}, \\ 3M - 3D_c & \text{if } 2D_c \le M \le 2D_c + D_d, \\ 3D_d + 3D_c & \text{if } 2D_c + D_d \le M. \end{cases}$$

VI. Proof of Theorem 3

Let $\bar{W}_i \triangleq \{W_{i+1}, W_{i+2}, \ldots, W_K\}$, $\bar{X}_i \triangleq \{\mathbf{x}_{i+1}^n, \mathbf{x}_{i+2}^n, \ldots, \mathbf{x}_K^n\}$, and $\bar{Y}_i \triangleq \{\mathbf{y}_{i+1}^n, \mathbf{y}_{i+2}^n, \ldots, \mathbf{y}_K^n\}$ $\forall i = 1, 2, \ldots, K$, where $\bar{W}_K = \bar{X}_K = \bar{Y}_K = \emptyset$. Starting from Fano's inequality, we have

$$\begin{aligned}
n&\left(\sum_{i=1}^K R_i - \epsilon_n\right) \\
&\le \sum_{i=1}^K I\left(W_i; \mathbf{y}_i^n\right) \\
&\overset{(a)}{\le} \sum_{i=1}^K I\left(W_i; \mathbf{y}_i^n, \bar{W}_i, \bar{Y}_i\right) \\
&\overset{(b)}{=} \sum_{i=1}^K I\left(W_i; \mathbf{y}_i^n, \bar{Y}_i | \bar{W}_i\right) \\
&= \sum_{i=1}^K h\left(\mathbf{y}_i^n, \bar{Y}_i | \bar{W}_i\right) - h\left(\mathbf{y}_i^n, \bar{Y}_i | \bar{W}_i, W_i\right) \\
&= \sum_{i=1}^K h\left(\mathbf{y}_i^n | \bar{Y}_i, \bar{W}_i\right) + \sum_{i=1}^K h\left(\bar{Y}_i | \bar{W}_i\right) - h\left(\mathbf{y}_i^n, \bar{Y}_i | \bar{W}_i, W_i\right) \\
&= \sum_{i=1}^K h\left(\mathbf{y}_i^n | \bar{Y}_i, \bar{W}_i\right) - h\left(\mathbf{y}_1^n, \mathbf{y}_2^n, \ldots, \mathbf{y}_K^n | W_1, W_2, \ldots, W_K\right) \\
&\quad + \sum_{i=2}^K h\left(\bar{Y}_i | \bar{W}_i\right) - h\left(\mathbf{y}_i^n, \bar{Y}_i | \bar{W}_i, W_i\right) \\
&\quad + h\left(\mathbf{y}_2^n, \ldots, \mathbf{y}_K^n | W_2, \ldots, W_K\right) \\
&\overset{(c)}{=} \sum_{i=1}^K h\left(\mathbf{y}_i^n | \bar{Y}_i, \bar{W}_i\right) - h\left(\mathbf{y}_1^n, \mathbf{y}_2^n, \ldots, \mathbf{y}_K^n | W_1, W_2, \ldots, W_K\right) \\
&\overset{(d)}{=} \sum_{i=1}^K h\left(\mathbf{y}_i^n \middle| \bar{Y}_i, \bar{W}_i, \bar{X}_i\right) - \sum_{i=1}^K \sum_{t=1}^n h\left(\mathbf{z}_i(t)\right) \\
&= \sum_{i=1}^K h\left(\sum_{j=1}^K \mathbf{H}_{i,j}\mathbf{x}_j^n + \mathbf{z}_i^n \middle| \bar{Y}_i, \bar{W}_i, \mathbf{x}_{i+1}^n, \mathbf{x}_{i+2}^n, \ldots, \mathbf{x}_K^n\right) \\
&\quad - \sum_{i=1}^K \sum_{t=1}^n h\left(\mathbf{z}_i(t)\right) \\
&= \sum_{i=1}^K h\left(\sum_{j=1}^i \mathbf{H}_{i,j}\mathbf{x}_j^n + \mathbf{z}_i^n \middle| \bar{Y}_i, \bar{W}_i, \bar{X}_i\right) - \sum_{i=1}^K \sum_{t=1}^n h\left(\mathbf{z}_i(t)\right) \\
&\overset{(e)}{=} \sum_{i=1}^K \sum_{t=1}^n h\Big(\sum_{j=1}^i \mathbf{H}_{i,j}\mathbf{x}_j(t) \\
&\qquad\qquad + \mathbf{z}_i(t) \Big| \mathbf{H}_{i,j}\mathbf{x}_j^{t-1} + \mathbf{z}_i^{t-1}, \bar{Y}_i, \bar{W}_i, \bar{X}_i\Big) \\
&\quad - \sum_{i=1}^K \sum_{t=1}^n h\left(\mathbf{z}_i(t)\right) \\
&\overset{(f)}{\le} \sum_{i=1}^K \sum_{t=1}^n h\left(\sum_{j=1}^i \mathbf{H}_{i,j}\mathbf{x}_j(t) + \mathbf{z}_i(t)\right) - n\sum_{i=1}^K h\left(\mathbf{z}_i(t)\right)
\end{aligned}$$

where (a) follows from the non-negativity of mutual information; (b) follows from the independence of $(W_1, W_2, \ldots, W_K)$; (c) follows from the recursive properties of $\bar{W}_i$ and $\bar{Y}_i$; (d) follows from the fact that $\mathbf{x}_i(t)$ is a function of $(W_i, \mathbf{y}_i^{t-1})$ and $\mathbf{x}_i^n$ is a function of $(W_i, \mathbf{y}_i^n)$; (e) follows from the chain rule for differential entropy; and (f) follows from the fact that conditioning reduces entropy.

Therefore, we have

$$\begin{aligned} n\left(\sum_{i=1}^{K} R_i - \epsilon_n\right) &\leq \sum_{i=1}^{K}\sum_{t=1}^{n} h\left(\sum_{j=1}^{i}\mathbf{H}_{i,j}\mathbf{x}_j(t) + \mathbf{z}_i(t)\right) - n\sum_{i=1}^{K} h\left(\mathbf{z}_i(t)\right) \\ &\leq \sum_{i=1}^{K}\sum_{t=1}^{n} h\left(\sum_{j=1}^{i}\mathbf{H}_{i,j}\mathbf{x}_j(t) + \mathbf{z}_i(t)\right) + no(\log(P)) \\ &\overset{(a)}{\leq} \sum_{i=1}^{K} n\Big((D_d + D_c(i-1)) \\ &\qquad + \log(P) + o(\log(P))\Big) + no(\log(P))\Big) \\ &= n\left(D_d K + \frac{K(K-1)D_c}{2}\right)\log(P) + no(\log(P)) \end{aligned}$$

where (a) follows from the fact that the pre-log term of $h(\sum_{j=1}^{i}\mathbf{H}_{i,j}\mathbf{x}_j(t) + \mathbf{z}_i(t))$ is constrained by $D_d + D_c(i-1)$, $\forall t = 1, 2, \ldots, n$. As a result, we get the following upper bound:

$$\Gamma_{fb} \leq \left(D_d K + \frac{K(K-1)D_c}{2}\right).$$

VII. Conclusion

In this paper, we have investigated the sum DoF of the K-user rank-deficient interference channel with feedback. When $K = 2$, we have developed an explicit achievable scheme and obtained a matching upper bound, thus completely characterizing the sum DoF. When $K = 3$, we have proposed a new achievable scheme which involves interference alignment, especially when the rank of cross links is sufficiently large as compared to the number of antennas at each node. In addition, we have derived an upper bound for the general K-user case.

We have showed that in contrast to the full-rank case, feedback can indeed increase the DoF by providing alternative signal paths. Furthermore, if we can use sufficiently many antennas at each node, this DoF gain increases proportionally with the number of users. Therefore, using feedback can be an attractive solution to overcome the rank-deficiency of channel matrices in a poor scattering environment.

Our work can be extended to several interesting directions: (1) Developing an achievable scheme for more general cases with respect to the number of antennas at each node and the rank of each channel matrix for $K \geq 3$; (2) Extending to other feedback models (e.g., limited feedback); (3) Extending to the cases where the channel state information at transmitters is either not available or delayed.

References

[1] C. E. Shannon, "The zero error capacity of a noisy channel," *IRE Trans. Inf. Theory*, vol. 2, no. 3, pp. 8–19, Sep. 1956.

[2] L. H. Ozarow, "The capacity of the white Gaussian multiple access channel with feedback," *IEEE Trans. Inf. Theory*, vol. 30, no. 4, pp. 623–629, Jul. 1984.

[3] A. El Gamal and Y.-H. Kim, *Network Information Theory*. Cambridge, U.K.: Cambridge Univ. Press, 2010.

[4] C. Suh and D. N. C. Tse, "Feedback capacity of the Gaussian interference channel to within 2 bits," *IEEE Trans. Inf. Theory*, vol. 57, no. 5, pp. 2667–2685, May 2011.

[5] C. Huang and S. A. Jafar, "Degrees of freedom of the MIMO interference channel with cooperation and cognition," *IEEE Trans. Inf. Theory*, vol. 55, no. 9, pp. 4211–4220, Sep. 2009.

[6] C. S. Vaze and M. K. Varanasi, "The degrees-of-freedom region of the MIMO interference channel with Shannon feedback," *IEEE Trans. Inf. Theory*, vol. 59, no. 8, pp. 4798–4810, Aug. 2013.

[7] I.-H. Wang and C. Suh, "Feedback increases the degrees of freedom of two unicast Gaussian networks," in *Proc. 50th Annu. Allerton Conf. Commun., Control, Comput.*, Monticello, IL, USA, Oct. 2012, pp. 1850–1855.

[8] D. Baum *et al.*, "WINNER deliverable 5.4 v.1.0, final report on link level and system level channel models," WINNER, New Delhi, India, Tech. Rep. IST-2003-507581, Nov. 2005.

[9] *IEEE 802.16m Evaluation Methodology Document (EMD)*, document IEEE 802.16m-08/004r2, Jul. 2008.

[10] J. Bicket, D. Aguayo, S. Biswas, and R. Morris, "Architecture and evaluation of an unplanned 802.11b mesh network," in *Proc. 11th Annu. Int. Conf. Mobile Comput., Netw., Commun. (MobiCom)*, Cologne, Germany, Aug. 2005, pp. 31–42.

[11] N. D. Skentos, A. G. Kanatas, and P. Constantinou, "MIMO channel characterization results from short range rooftop to rooftop wideband measurements," in *Proc. IEEE Int. Conf. Wireless Mobile Comput., Netw. Commun. (WiMob)*, Cologne, Germany, Aug. 2005, pp. 137–144.

[12] S. H. Chae and S.-Y. Chung, "On the degrees of freedom of rank deficient interference channels," in *Proc. IEEE Int. Symp. Inf. Theory*, Saint Petersburg, Russia, Jul./Aug. 2011, pp. 1367–1371.

[13] S. R. Krishnamurthy and S. A. Jafar, "Degrees of freedom of 2-user and 3-user rank-deficient MIMO interference channels," in *Proc. IEEE Global Telecommun. Conf. (GLOBECOM)*, Anaheim, CA, USA, Dec. 2012, pp. 2462–2467.

[14] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*. Cambridge, U.K.: Cambridge Univ. Press, 2005.

[15] S. H. Chae, S.-W. Jeon, and S.-Y. Chung, "Cooperative relaying for the rank-deficient MIMO relay interference channel," *IEEE Commun. Lett.*, vol. 16, no. 1, pp. 9–11, Jan. 2012.

[16] F. D. Neeser and J. L. Massey, "Proper complex random processes with applications to information theory," *IEEE Trans. Inf. Theory*, vol. 39, no. 4, pp. 1293–1302, Jul. 1993.

[17] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. New York, NY, USA: Wiley, 2005.

[18] V. R. Cadambe and S. A. Jafar, "Interference alignment and degrees of freedom of the K-user interference channel," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3425–3441, Aug. 2008.

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