To Feedback or Not to Feedback

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***Abstract*—We explore two-way interference channels (ICs) where there are forward and backward ICs with four independent messages: two associated with the forward IC and the other two with respect to the backward IC. For a linear deterministic model of this channel, we develop inner and outer bounds on the capacity region. As a consequence, we demonstrate that interaction across forward and backward channels enables a more beneficial use of the channels, thereby yielding strict capacity improvements over non-interactive independent transmission. Moreover, our novel outer bound establishes the characterization of channel regimes in which interaction has no bearing on sum capacity.**

I. Introduction

The inherent two-way nature of communication links provides an opportunity to enable *interaction* between nodes. It allows nodes to adapt their transmitted signals to past received signals in exchanging their messages. Interaction at a node is enabled through the use of its past received signals, which are usually obtained via *feedback* offered by backward communication links associated with that node. Hence, exploring the role of feedback needs to be preceded towards understanding two-way communication.

The history of feedback traces back to Shannon [1]. For memoryless point-to-point channels, Shannon proved that feedback cannot increase capacity [1], but subsequent work showed that the situation is different for many multi-user channels. In particular, recent research shows that feedback provides a significant gain for communication over interference channels (ICs) [2], [3], [4]. Interestingly an explicit analysis in [3] reveals that the feedback gain is unbounded, i.e., the gap between the feedback and nonfeedback capacities can be arbitrarily large for certain channel parameters.

While feedback promises substantial theoretical gain, it comes with challenges in implementation. The reason is that the feedback gain analyzed in [3] is concerning an idealistic scenario in which the channel output feedback is given for free. In the scenario, the cost of using feedback is not taken into account and thus the theoretical feedback gain therein is not promised yet in practice. So one natural question that arises is: Can feedback provide a gain even when the feedback cost is taken into consideration?

In an effort to address this question, we consider two-way interference channels where there are forward and backward ICs (not necessarily the same) with four independent messages: two associated with the forward IC and the other two with respect to the backward IC. In the considered model, two transmissions compete for the use of each channel, e.g., the backward IC: (1) Sending independent backward messages; (2) Sending *feedback* signals to aid forward-message transmission. So in this model, feedback cost is well reflected via the tension between the two transmissions. As a stepping stone towards practically relevant Gaussian channels, we use an intermediate model: the Avestimehr-Diggavi-Tse (ADT) deterministic model [5], which is known to well capture key properties of the wireless Gaussian channels.

Our contributions are two-folded. We first characterize an inner bound on the capacity region of the two-way IC. As a result, we demonstrate that interaction between forward and backward ICs can provide capacity improvements over non-interactive transmission. The second contribution is the derivation of a novel outer bound, which one cannot obtain with prior techniques, such as the cutset bound, the genie-aided bound [6], [2], [3] and the generalized network sharing bound [7]. This new development leads to the characterization of entire channel regimes in which interaction provides no gain in sum capacity.

Related Work: The most related works are Sahai *et.al.* [8] and Suh-Wang-Tse [9]. Sahai *et.al.* [8] demonstrated that there is no interaction gain when forward-and-backward channels are identical and lie in the strong interference regime. Suh-Wang-Tse [9] established more broader channel regimes where interaction provides no gain, as well as identified some other regimes in which interaction offers a gain. However, the complete characterization of the regimes with gain or without gain has been open. Our main contribution of this paper is to *settle this open problem*. We characterize channel regimes with gain or without gain, under the channel model considered in [8], [9] and herein, thereby obtaining the gain-vs-nogain picture (see Fig. 2). A key innovation behind this result is the development of a novel outer bound. On the other hand, Cheng-Devroye [10] considered a partial two-way transmission scenario where interaction is enabled only at two nodes, while no interaction is permitted at the other two nodes. It has been shown under the scenario that the partial interaction provides no gain for some channel regimes.

II. Model

Fig. 1 describes a two-way ADT deterministic IC where user k wants to send its own message W_k to user $\tilde{k}$, while user $\tilde{k}$

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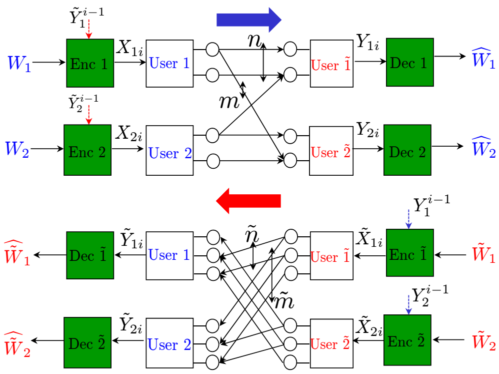

Fig. 1. Two-way ADT deterministic interference channel (IC).

wishes to send its own message $\tilde{W}_k$ to user k, $k = 1, 2$. We assume that $(W_1, W_2, \tilde{W}_1, \tilde{W}_2)$ are independent and uniformly distributed. For simplicity, we consider a setting where both forward and backward ICs are symmetric but not necessarily the same. In the forward IC, n and m indicate the number of signal bit levels for direct and cross links respectively. The corresponding values in the backward IC are denoted by $(\tilde{n}, \tilde{m})$. Let $X_k \in \mathbb{F}_2^{\max(n,m)}$ be user k's transmitted signal and $V_k \in \mathbb{F}_2^m$ be a part of X_k visible to user $\tilde{j}(\neq \tilde{k})$. Similarly let $\tilde{X}_k$ be user $\tilde{k}$'s transmitted signal and $\tilde{V}_k$ be a part of $\tilde{X}_k$ visible to user $j(\neq k)$. The deterministic model captures broadcast and superposition of signals. See [5] for explicit details. A signal bit level observed by both users is broadcasted. If multiple signal levels arrive at the same signal level at a user, we assume modulo-2-addition.

The encoded signal X_{ki} of user k at time i is a function of its own message and past received signals: $X_{ki} = f_{ki}(W_k, \tilde{Y}_k^{i-1})$. We define $\tilde{Y}_k^{i-1} := \{\tilde{Y}_{kt}\}_{t=1}^{i-1}$ where $\tilde{Y}_{kt}$ denotes user k's received signal at time t, offered through the backward IC. Similarly the encoded signal $\tilde{X}_{ki}$ of user $\tilde{k}$ at time i is a function of its own message and past received signals: $\tilde{X}_{ki} = \tilde{f}_{ki}(\tilde{W}_k, Y_k^{i-1})$.

A rate tuple $(R_1, R_2, \tilde{R}_1, \tilde{R}_2)$ is said to be achievable if there exists a family of codebooks and encoder/decoder functions such that the decoding error probabilities go to zero as code length N tends to infinity. In this work, we focus on a sum-rate pair $(R, \tilde{R}) := (R_1 + R_2, \tilde{R}_1 + \tilde{R}_2)$ and the corresponding capacity region, defined as the closure of the set of achievable sum-rate pairs: $\mathcal{C} = \text{closure}\{(R, \tilde{R}) : (R_1, R_2, \tilde{R}_1, \tilde{R}_2) \in \mathcal{C}_{\text{high}}\}$ where $\mathcal{C}_{\text{high}}$ denotes the capacity region w.r.t. the high-dimensional rate tuple $(R_1, R_2, \tilde{R}_1, \tilde{R}_2)$. Let $C^{\text{sum}} = \sup\{R + \tilde{R} : (R, \tilde{R}) \in \mathcal{C}\}$.

III. Main Results

We first state the capacity region for the non-interactive scenario, which we will use as a baseline for comparisons to our main results that will be stated later.

Theorem 1 (Non-interaction capacity [6], [11]): The capacity region $\mathcal{C}_{\text{no}}$ for the non-interactive scenario is the set of $(R, \tilde{R})$ such that $R \le C_{\text{no}}$ and $\tilde{R} \le \tilde{C}_{\text{no}}$ where

$$C_{\text{no}} = \min\{2\max(n-m, m), \max(2n-m, m), 2n\}, \quad (1)$$
$$\tilde{C}_{\text{no}} = \min\{2\max(\tilde{n}-\tilde{m}, \tilde{m}), \max(2\tilde{n}-\tilde{m}, \tilde{m}), 2\tilde{n}\}. \quad (2)$$

A. Inner Bound

Theorem 2 (Inner bound): Let $\alpha = \frac{m}{n}$ and $\tilde{\alpha} = \frac{\tilde{m}}{\tilde{n}}$. The capacity region of the two-way IC includes the set $\mathcal{R}$ of $(R, \tilde{R})$ such that, for some $0 \le \lambda \le 1$ and $0 \le \tilde{\lambda} \le 1$,

$$R \le \begin{cases} (1-\lambda)\min\left\{C_{\text{no}} + \frac{2\tilde{\lambda}}{1-\lambda}\max(\tilde{n}-\tilde{m}, \tilde{m}), C_{\text{pf}}\right\}, & \alpha < \frac{2}{3}, \\ (1-\lambda)C_{\text{no}}, & \frac{2}{3} \le \alpha < 2, \\ (1-\lambda)\min\left\{C_{\text{no}} + \frac{2\tilde{\lambda}\tilde{n}}{1-\lambda}, C_{\text{pf}}\right\}, & \alpha \ge 2, \end{cases}$$

$$\tilde{R} \le \begin{cases} (1-\tilde{\lambda})\min\left\{\tilde{C}_{\text{no}} + \frac{2\lambda}{1-\tilde{\lambda}}\max(n-m, m), \tilde{C}_{\text{pf}}\right\}, & \tilde{\alpha} < \frac{2}{3}, \\ (1-\tilde{\lambda})\tilde{C}_{\text{no}}, & \frac{2}{3} \le \tilde{\alpha} < 2, \\ (1-\tilde{\lambda})\min\left\{\tilde{C}_{\text{no}} + \frac{2\lambda n}{1-\tilde{\lambda}}, \tilde{C}_{\text{pf}}\right\}, & \tilde{\alpha} \ge 2. \end{cases}$$

Here C_{pf} and $\tilde{C}_{\text{pf}}$ indicate the perfect-feedback sum capacities of the forward and backward ICs respectively [3]: $C_{\text{pf}} = \max(2n-m, m)$; $\tilde{C}_{\text{pf}} = \max(2\tilde{n}-\tilde{m}, \tilde{m})$.

Proof: See Section IV. ■

Remark 1: The inner bound formula is closely coupled with our achievability built upon a particular frequency band coordination in which a band assigned to each channel is split into two parts: (1) one for its own message transmission; (2) the other for feedback transmission. Here a parameter $\lambda \in [0, 1]$ represents the fraction of feedback transmission in the forward channel. Similarly $\tilde{\lambda}$ is defined w.r.t. the backward channel. This will be clearer in Section IV with Fig. 3. □

Theorems 1 and 2 allow us to check whether or not there is a gain due to interaction. Specifically we make an explicit comparison by evaluating the μ-sum rates:

$$\max_{(R,\tilde{R})\in\mathcal{C}_{\text{no}}} R + \mu\tilde{R} \overset{?}{<} \max_{(R,\tilde{R})\in\mathcal{R}} R + \mu\tilde{R}.$$

Here one can interpret the μ as the ratio of the revenue of the backward rate $\tilde{R}$ to that of R. For instance, $\mu = 0$ represents the case where there is no revenue reaped by transmissions intended for backward-message delivery; $\mu = 1$ indicates the case in which the revenues are the same for both transmissions. Strict inequality for some μ in the above implies that there exits a revenue scenario in which interaction provides a gain over non-interactive transmission. Here we examine perhaps the most interesting and practically-relevant scenario: $\mu = 1$, in which the revenues w.r.t. R and $\tilde{R}$ are the same, and hence any interaction gain that can be reaped in the case can be viewed as a *net* gain.

Comparison 1 (Net Gain): For illustration purpose, we consider an example of $(n, m) = (2, 1)$ and $(\tilde{n}, \tilde{m}) = (1, 1)$. In this example, Theorem 1 gives $C_{\text{no}}^{\text{sum}} = C_{\text{no}} + \tilde{C}_{\text{no}} = 2 + 1 = 3$, while Theorem 2 yields:

$$R^{\text{sum}} = \max_{0\le\lambda,\tilde{\lambda}\le 1} \min\left\{2(1-\lambda) + 2\tilde{\lambda}, 3(1-\lambda)\right\} + 1 - \tilde{\lambda}$$
$$= 3.5 > 3 = C_{\text{no}}^{\text{sum}}$$

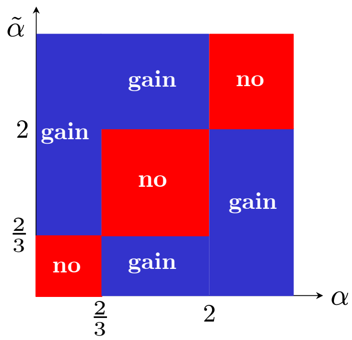

Fig. 2. Gain-vs-nogain picture: The blue regime indicates the case in which $R^{\text{sum}} > C_{\text{no}}^{\text{sum}}$, while the red regime denotes the case where $\bar{C}^{\text{sum}} = C_{\text{no}}^{\text{sum}}$.

where $R^{\text{sum}} = 3.5$ is achieved when $(\lambda, \tilde{\lambda}) = (0, 0.5)$. Notice that interaction provides around 16.7% net gain over non-interactive transmission. Later in Section IV (particularly see Remark 3), we will provide an intuition as to where this net gain comes from. In Corollary 1 (see below), we also identify other channel regimes in which interaction provides capacity improvements. See Fig. 2. □

Corollary 1 ($\mu = 1$): $R^{\text{sum}} > C_{\text{no}}^{\text{sum}}$ for the regimes: (G1) $(\alpha < \frac{2}{3}, \tilde{\alpha} > \frac{2}{3})$; (G2) $(\frac{2}{3} < \alpha < 2, \tilde{\alpha} < \frac{2}{3})$; (G3) $(\frac{2}{3} < \alpha < 2, \tilde{\alpha} > 2)$; and (G4) $(\alpha \geq 2, \tilde{\alpha} < 2)$.

Proof: A tedious yet straightforward computation with Theorems 1 and 2 completes the proof. ■

B. Outer Bound

The identification of regimes that exhibit a net gain, summarized in Corollary 1, is based on an inner bound in Theorem 2, which is due to a particular achievable scheme that will be described in Section IV. So a natural question that arises is: Can interaction offer a net gain also for the other remaining regimes outside (G1)-(G4)? In other words, is there another achievable scheme that enables a net gain for the remaining regimes? We answer this question *negatively* by establishing a novel outer bound as follows.

Theorem 3 (Outer Bound): The capacity region of the two-way IC is included by the set $\bar{\mathcal{C}}$ of $(R, \tilde{R})$ such that

$$R \leq (n-m)^+ + \max(n, m) \quad (3)$$

$$\tilde{R} \leq (\tilde{n}-\tilde{m})^+ + \max(\tilde{n}, \tilde{m}) \quad (4)$$

$$R + \tilde{R} \leq 2(n + \tilde{n}) \quad (5)$$

$$R + \tilde{R} \leq 2\max(n-m, m) + 2\max(\tilde{n}-\tilde{m}, \tilde{m}). \quad (6)$$

Proof: Notice that the first two bounds match with the perfect-feedback bound [3], [8], [9]. So one can prove them with a simple modification to the proof in the references. The third bound comes from the following cutset bounds, of which the proofs will be omitted here: $R_1 + \tilde{R}_2 \leq n + \tilde{n}$ and $R_2 + \tilde{R}_1 \leq n + \tilde{n}$. Our key contribution lies in the derivation of the last bound, which will be provided in Section V. ■

Comparison 2 (No Gain): The bounds of (3) and (4) in Theorem 3 reveals that for the regime of $(\frac{2}{3} \leq \alpha \leq 2, \frac{2}{3} \leq \tilde{\alpha} \leq 2)$, $R + \tilde{R} \leq 2n - m + 2\tilde{n} - \tilde{m}$. This bound coincides with $C_{\text{no}} + \tilde{C}_{\text{no}}$, suggesting no gain. Similarly using the bound of (5), one can show that interaction does not help when $(\alpha \geq 2, \tilde{\alpha} \geq 2)$. The last regime of $(\alpha < \frac{2}{3}, \tilde{\alpha} < \frac{2}{3})$ has been open thus far. Our main contribution is in the characterization of the open regime with the novel bound (6). Note that this bound yields $R + \tilde{R} \leq 2\max(n-m, m) + 2\max(\tilde{n}-\tilde{m}, \tilde{m})$ equal to $C_{\text{no}}^{\text{sum}}$, implying no gain. See Fig. 2. □

Remark 2: Our inner and upper bounds do not match in general. For instance, a careful inspection reveals that our achievable sum rate due to Theorem 2 is strictly less than the upper bound in Theorem 3 except for the no-gain regimes in Fig. 2. □

IV. Proof of Theorem 2

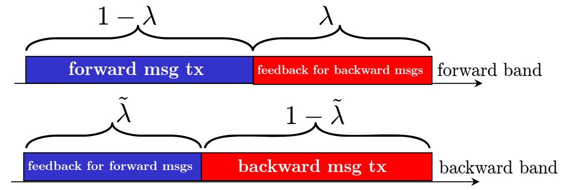

Fig. 3. Proposed scheme: Coordination of forward and backward bands.

Our proposed scheme employs the following frequency band coordination. We split a band assigned to each channel into two parts: one for its own message transmission and the other for feedback that aids the other message transmission w.r.t. the other channel. See Fig. 3.

Under this coordination, each channel is then used either for its own message transmission or for feedback w.r.t. messages associated with the other cannel. In other words, fresh message transmission and feedback transmission are *orthogonalized* each other. This orthogonality ensures that the signals in the blue colored bands (consisting of the $(1-\lambda)$ fraction of the forward band and the $\tilde{\lambda}$ fraction of the backward band in Fig. 3) contain forward-message related signals only, and hence R is determined solely by transmission that occurs in the blue colored bands. Similarly transmission through the red colored bands contributes to $\tilde{R}$.

We are now ready to prove the claimed inner bound. First consider the forward-message sum rate. We normalize the fraction $(1-\lambda)$ of the forward band assigned for forward-message transmission to 1, and let $\gamma = \frac{\tilde{\lambda}}{1-\lambda}$. Then, R would be $(1-\lambda)$ times a sum rate for the case in which the forward and backward symbol rates are 1 and γ respectively. In fact, an achievable rate in the normalized setting is derived in [9], which is formally stated in Lemma 1 (see below). Using Lemma 1, one can readily obtain the claimed achievable rate. Similarly one can prove the achievable rate w.r.t. $\tilde{R}$, which completes the proof.

Lemma 1 ([9]): Suppose that the forward and backward symbol rates are 1 and γ respectively. Then, one can achieve

$$R = \begin{cases} \min\{C_{\text{no}} + 2\gamma\max(\tilde{n}-\tilde{m}, \tilde{m}), C_{\text{pf}}\}, & \alpha < \frac{2}{3}, \\ \min\{C_{\text{no}} + 2\gamma\tilde{n}, C_{\text{pf}}\}, & \alpha \geq 2. \end{cases}$$

Proof: See [9, Section IV] for the proof. ■

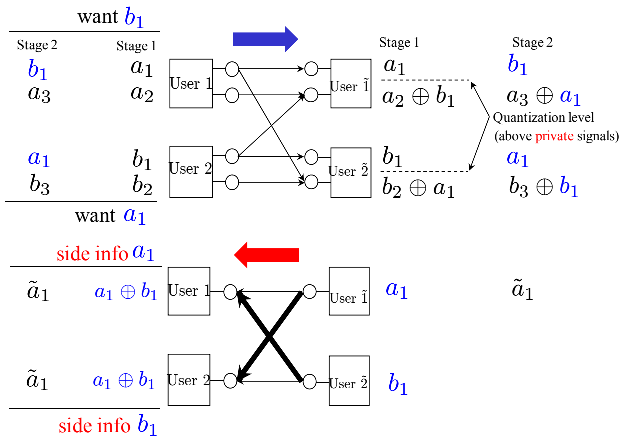

Fig. 4. An achievable scheme for $\alpha := \frac{m}{n} = \frac{1}{2}$, $\tilde{\alpha} := \frac{\tilde{m}}{\tilde{n}} = 1$: $R^{\text{sum}} = 3 + 0.5 > 3 = C_{\text{no}}^{\text{sum}}$ is achieved when $(\lambda, \tilde{\lambda}) = (0, 0.5)$.

A. Example: $(n, m) = (2, 1), (\tilde{n}, \tilde{m}) = (1, 1)$

In this section, we seek to illustrate achievability for the example considered in Comparison 1, as an effort to provide an intuition behind the nature of the net gain that we claimed earlier. Notice in Corollary 1 that in the considered example, we achieve $R^{\text{sum}} = 3 + 0.5 > 3 = C_{\text{no}}^{\text{sum}}$ when $(\lambda, \tilde{\lambda}) = (0, 0.5)$.

Let us first see how we achieve $R = 3$. Similar to the perfect feedback scheme [3], it has two stages. See Fig. 4. In the first stage, each user sends one private bit and one common bit on the two levels. In the perfect feedback scheme, user 1 wanted to know the other user information b_1 which caused interference to its desired symbol a_2. Similarly user 2 wanted to know a_1. In an attempt to satisfy this demand, the two bits $(a_2 \oplus b_1, b_2 \oplus a_1)$ were fed back to the users. It appears that two time slots are needed to feed back these two bits. However, one can satisfy the demand in one shot. Note that the symbol b_1 wanted by user 1 is available at user $\tilde{2}$. Similarly the symbol a_1 wanted by user 2 is available at user $\tilde{1}$. Suppose we now send these two bits instead. Users 1 and 2 can then decode b_1 and a_1 respectively, exploiting its own signal as side information. The key observation here is that exploiting side information at users 1 and 2, the backward IC becomes equivalent to *two non-interfering cross point-to-point channels*. In the second stage, each user starts with sending one fresh private bit on the bottom level, and additionally sends the other user's information (decoded with feedback) on the vacant common level. Users $\tilde{1}$ and $\tilde{2}$ can then decode 6 bits in total during the two stages, thus achieving $R = 3$.

Observe that the forward channel is fully utilized only for forward-message transmission, i.e., $1 - \lambda = 1$, while the backward channel is utilized once every two slots for feedback, i.e., $\tilde{\lambda} = 0.5$. On the other hand, the non-interactive backward-message transmission can be made for the remaining $(1 - \tilde{\lambda})$-fraction of the backward band, thereby yielding $\tilde{R} = 0.5$. Hence, $R^{\text{sum}} = 3 + 0.5 = 3.5$, which shows around 16.7% net gain over non-interaction transmission with $C_{\text{no}}^{\text{sum}} = 3$.

Remark 3 (Why Net Gain?): Note in Fig. 4 that the two bits (a_1, b_1) can be successfully fed back through the one-bit-capacity backward IC. This is because each user can cancel the seemingly interfering information by exploiting its own information as side information. This enables the effective feedback gain: a capacity increase of 1 bit with $\tilde{\lambda}\tilde{C}_{\text{no}} = 1/2$ bits of the backward IC's original capability. This is the very reason as to why the net gain occurs. □

V. Proof of Theorem 3

We focus on the proof of the novel bound (6) which hinges upon several lemmas stated below. The proof is streamlined with the help of a key notion, called *triple mutual information* (or interaction information [12]), which is defined as $I(X;Y;Z) := I(X;Y) - I(X;Y|Z)$. It turns out that the commutative property of the notion plays a crucial role in deriving several key steps in the proof: $I(X;Y;Z) = I(X;Z;Y) = \cdots = I(Z;Y;X)$. Using this notion and starting with Fano's inequality, we get

$$
\begin{aligned}
&N(R_1 + R_2 - \epsilon_N) \le I(W_1; Y_1^N, \tilde{W}_1) + I(W_2; Y_2^N, \tilde{W}_2) \\
&\le I(W_1; Y_1^N, V_1^N | \tilde{W}_1) + I(W_2; Y_2^N, V_2^N | \tilde{W}_2) \\
&= \sum \Big\{ I(W_1; Y_{1i}, V_{1i} | \tilde{W}_1, Y_1^{i-1}, V_1^{i-1}) \\
&\qquad + I(W_2; Y_{2i}, V_{2i} | \tilde{W}_2, Y_2^{i-1}, V_2^{i-1}) \Big\} \\
&= \sum \Big\{ I(V_{1i}; W_1 | \tilde{W}_1, Y_1^{i-1}, V_1^{i-1}) + I(Y_{1i}; W_1 | \tilde{W}_1, Y_1^{i-1}, V_1^{i}) \\
&\quad + I(V_{2i}; W_2 | \tilde{W}_2, Y_2^{i-1}, V_2^{i-1}) + I(Y_{2i}; W_2 | \tilde{W}_2, Y_2^{i-1}, V_2^{i}) \Big\} \\
&\overset{(a)}{=} \sum \Big\{ I(Y_{1i}; W_1, W_2, \tilde{W}_2 | \tilde{W}_1, Y_1^{i-1}, V_1^{i}) \\
&\qquad + I(Y_{2i}; W_2, W_1, \tilde{W}_1 | \tilde{W}_2, Y_2^{i-1}, V_2^{i}) \\
&+ I(V_{1i}; W_1 | \tilde{W}_1, Y_1^{i-1}, V_1^{i-1}) - I(Y_{1i}; W_2, \tilde{W}_2 | W_1, \tilde{W}_1, Y_1^{i-1}, V_1^{i}) \\
&+ I(V_{2i}; W_2 | \tilde{W}_2, Y_2^{i-1}, V_2^{i-1}) - I(Y_{2i}; W_1, \tilde{W}_1 | W_2, \tilde{W}_2, Y_2^{i-1}, V_2^{i}) \Big\} \\
&\le \sum \{ H(Y_{1i}|V_{1i}) + H(Y_{2i}|V_{2i}) \\
&+ I(V_{1i}; W_1 | \tilde{W}_1, Y_1^{i-1}, V_1^{i-1}) - I(Y_{1i}; W_2, \tilde{W}_2 | W_1, \tilde{W}_1, Y_1^{i-1}, V_1^{i}) \\
&+ I(V_{2i}; W_2 | \tilde{W}_2, Y_2^{i-1}, V_2^{i-1}) - I(Y_{2i}; W_1, \tilde{W}_1 | W_2, \tilde{W}_2, Y_2^{i-1}, V_2^{i}) \Big\}
\end{aligned}
$$

where (a) follows from a chain rule. Now using Lemma 2 (see below), we get:

$$
\begin{aligned}
&N(R_1 + R_2 + \tilde{R}_1 + \tilde{R}_2 - \epsilon_N) \\
&\le \sum \Big\{ H(Y_{1i}|V_{1i}) + H(Y_{2i}|V_{2i}) + H(\tilde{Y}_{1i}|\tilde{V}_{1i}) + H(\tilde{Y}_{2i}|\tilde{V}_{2i}) \Big\} \\
&\le 2N \max(n - m, m) + 2N \max(\tilde{n} - \tilde{m}, \tilde{m}).
\end{aligned}
$$

If $(R_1, R_2, \tilde{R}_1, \tilde{R}_2)$ is achievable, then $\epsilon_N \to 0$ as N tends to infinity. Therefore, we get the desired bound.

Lemma 2:

$$
\begin{aligned}
&\sum \Big\{ I(V_{1i}; W_1 | \tilde{W}_1, Y_1^{i-1}, V_1^{i-1}) - I(Y_{1i}; W_2, \tilde{W}_2 | W_1, \tilde{W}_1, Y_1^{i-1}, V_1^{i}) \\
&\quad + I(V_{2i}; W_2 | \tilde{W}_2, Y_2^{i-1}, V_2^{i-1}) - I(Y_{2i}; W_1, \tilde{W}_1 | W_2, \tilde{W}_2, Y_2^{i-1}, V_2^{i}) \\
&\quad + I(\tilde{V}_{1i}; \tilde{W}_1 | W_1, \tilde{Y}_1^{i-1}, \tilde{V}_1^{i-1}) - I(\tilde{Y}_{1i}; \tilde{W}_2, W_2 | \tilde{W}_1, W_1, \tilde{Y}_1^{i-1}, \tilde{V}_1^{i}) \\
&\quad + I(\tilde{V}_{2i}; \tilde{W}_2 | W_2, \tilde{Y}_2^{i-1}, \tilde{V}_2^{i-1}) - I(\tilde{Y}_{2i}; \tilde{W}_1, W_1 | \tilde{W}_2, W_2, \tilde{Y}_2^{i-1}, \tilde{V}_2^{i}) \Big\} \\
&\le 0
\end{aligned}
$$

Proof: Consider 1st and 2nd terms in summation of LHS:

$$\begin{aligned}
&\overset{(a)}{=} \sum \Big\{ I(V_{1i}; W_1 | \tilde{W}_1, Y_1^{i-1}, V_1^{i-1}) \\
&\quad - I(Y_{1i}; W_2, \tilde{W}_2, \tilde{Y}_1^i | W_1, \tilde{W}_1, Y_1^{i-1}, V_1^i) \\
&\overset{(b)}{=} \sum \Big\{ I(V_{1i}, \tilde{V}_{1i}; W_1 | \tilde{W}_1, Y_1^{i-1}, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\quad - I(Y_{1i}; \tilde{Y}_1^i | W_1, \tilde{W}_1, Y_1^{i-1}, V_1^i, \tilde{V}_1^i) \\
&\quad - I(Y_{1i}; W_2, \tilde{W}_2 | W_1, \tilde{W}_1, \tilde{Y}_1^{i-1}, Y_1^{i-1}) \Big\} \\
&= \sum \Big\{ I(V_{1i}, \tilde{V}_{1i}; W_1 | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\quad - I(V_{1i}, \tilde{V}_{1i}; W_1; Y_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\quad - I(Y_{1i}; \tilde{Y}_1^i | W_1, \tilde{W}_1, Y_1^{i-1}, V_1^i, \tilde{V}_1^i) \\
&\quad - I(Y_{1i}; W_2, \tilde{W}_2 | W_1, \tilde{W}_1, \tilde{Y}_1^i, Y_1^{i-1}) \Big\}
\end{aligned}$$

where (a) follows from the fact that $\tilde{Y}_1^i$ is a function of $(W_1, \tilde{W}_1, W_2, \tilde{W}_2)$ and (b) follows from the fact that $\tilde{V}_1^i$ is a function of $(\tilde{W}_1, Y_1^{i-1})$.

Using Lemma 3 stated below and applying the same to 5th and 6th terms in summation of LHS, we get:

(1st, 2nd, 5th and 6th terms of LHS in the claimed bound)

$$\begin{aligned}
&\overset{(a)}{\leq} \sum \Big\{ I(V_{1i}, \tilde{V}_{1i}; W_1 | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\quad + I(\tilde{Y}_{1i}; Y_1^{i-1} | W_1, \tilde{W}_1, \tilde{Y}_1^{i-1}, \tilde{V}_1^i) \\
&\quad - I(\tilde{V}_{1i}; W_1, \tilde{Y}_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\quad - I(Y_{1i}; W_2, \tilde{W}_2 | W_1, \tilde{W}_1, \tilde{Y}_1^i, Y_1^{i-1}) \Big\} \\
&+ \sum \Big\{ I(\tilde{V}_{1i}; \tilde{W}_1 | W_1, \tilde{Y}_1^{i-1}, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\quad - I(\tilde{Y}_{1i}; \tilde{W}_2, W_2, Y_1^{i-1} | \tilde{W}_1, W_1, \tilde{Y}_1^{i-1}, \tilde{V}_1^i) \Big\} \\
&\overset{(b)}{\leq} \sum \Big\{ I(V_{1i}, \tilde{V}_{1i}; W_1 | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\quad - I(\tilde{V}_{1i}; W_1, \tilde{Y}_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\quad - I(Y_{1i}; W_2, \tilde{W}_2 | W_1, \tilde{W}_1, \tilde{Y}_1^i, Y_1^{i-1}) \Big\} \\
&+ \sum \Big\{ I(\tilde{V}_{1i}; \tilde{W}_1, W_1, \tilde{Y}_1^{i-1} | V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\quad - I(\tilde{Y}_{1i}; \tilde{W}_2, W_2 | \tilde{W}_1, W_1, \tilde{Y}_1^{i-1}, Y_1^{i-1}) \Big\} \\
&\overset{(c)}{\leq} \sum \Big\{ I(V_{1i}, \tilde{V}_{1i}; W_1 | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\quad - I(Y_{1i}, \tilde{Y}_{1i}; W_2, \tilde{W}_2 | W_1, \tilde{W}_1, \tilde{Y}_1^{i-1}, Y_1^{i-1}) \\
&\quad + I(V_{1i}, \tilde{V}_{1i}; \tilde{W}_1 | V_1^{i-1}, \tilde{V}_1^{i-1}) \Big\} \\
&\overset{(d)}{=} I(V_1^N, \tilde{V}_1^N; W_1, \tilde{W}_1) - I(Y_1^N, \tilde{Y}_1^N, V_2^N, \tilde{V}_2^N; W_2, \tilde{W}_2 | W_1, \tilde{W}_1) \\
&\leq I(V_1^N, \tilde{V}_1^N; W_1, \tilde{W}_1) - I(V_2^N, \tilde{V}_2^N; W_2, \tilde{W}_2 | W_1, \tilde{W}_1)
\end{aligned}$$

where (a) follows from the fact that V_1^{i-1} and Y_1^{i-1} are functions of $(W_1, \tilde{Y}_1^{i-1})$ and $(W_1, W_2, \tilde{W}_1, \tilde{W}_2)$, respectively; (b) follows from a chain rule (applied on the last term) and the non-negativity of mutual information; (c) follows from a chain rule and the non-negativity of mutual information; (d) follows from a chain rule and the fact that $(V_2^N, \tilde{V}_2^N)$ is a function of $(W_1, \tilde{W}_1, Y_1^N, \tilde{Y}_1^N)$.

Applying the same to 3rd, 4th, 7th and 8th terms in summation of LHS, we get:

(LHS in the claimed bound)

$$\begin{aligned}
&\leq I(W_2, \tilde{W}_2, V_1^N, \tilde{V}_1^N; W_1, \tilde{W}_1) - I(V_2^N, \tilde{V}_2^N; W_2, \tilde{W}_2 | W_1, \tilde{W}_1) \\
&+ I(W_1, \tilde{W}_1, V_2^N, \tilde{V}_2^N; W_2, \tilde{W}_2) - I(V_1^N, \tilde{V}_1^N; W_1, \tilde{W}_1 | W_2, \tilde{W}_2) \\
&= 0.
\end{aligned}$$

Lemma 3:

$$\begin{aligned}
&- \sum \Big\{ I(V_{1i}, \tilde{V}_{1i}; W_1; Y_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad + I(Y_{1i}; \tilde{Y}_1^i | W_1, \tilde{W}_1, Y_1^{i-1}, V_1^i, \tilde{V}_1^i) \Big\} \\
&\leq \sum \Big\{ I(\tilde{Y}_{1i}; Y_1^{i-1} | W_1, \tilde{W}_1, \tilde{Y}_1^{i-1}, \tilde{V}_1^i) \\
&\qquad - I(\tilde{V}_{1i}; W_1, \tilde{Y}_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \Big\}.
\end{aligned}$$

Proof: See Appendix A. ■

■

VI. Conclusion

In this work, we established inner and outer bounds of the two-way ADT deterministic IC, thereby categorizing channel regimes into (i) one where interaction provides a gain in sum capacity and (ii) the other in which it does not. Our achievability does not fully utilize interaction because it does not allow mixing feedback with pure message signals. Hence one future work of interest would be to characterize the capacity region of the two-way IC to identify the entire channel regime in which stronger interaction that permits blending the signals offers a larger gain.

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Appendix A
Proof of Lemma 3

$$
\begin{aligned}
&- \sum \Big\{ I(V_{1i}, \tilde{V}_{1i}; W_1; Y_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad + I(Y_{1i}; \tilde{Y}_1^i | W_1, \tilde{W}_1, Y_1^{i-1}, V_1^i, \tilde{V}_1^i) \Big\} \\
&\overset{(a)}{=} \sum \Big\{ I(V_{1i}, \tilde{V}_{1i}; Y_1^{i-1} | W_1, \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad - I(V_{1i}, \tilde{V}_{1i}; Y_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad - I(Y_{1i}; \tilde{Y}_1^i | W_1, \tilde{W}_1, Y_1^{i-1}, V_1^i, \tilde{V}_1^i) \Big\} \\
&\overset{(b)}{\le} \sum \Big\{ I(V_{1i}, \tilde{V}_{1i}; Y_1^{i-1} | W_1, \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad - I(\tilde{V}_{1i}; Y_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad - I(Y_{1i}; \tilde{Y}_1^i | W_1, \tilde{W}_1, Y_1^{i-1}, V_1^i, \tilde{V}_1^i) \Big\} \\
&\overset{(c)}{=} \sum \Big\{ I(\tilde{Y}_1^i, \tilde{V}_{1i}; Y_1^{i-1} | W_1, \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad - I(\tilde{Y}_1^i; Y_1^{i-1} | W_1, \tilde{W}_1, V_1^i, \tilde{V}_1^i) \\
&\qquad - I(\tilde{V}_{1i}; Y_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad - I(Y_{1i}; \tilde{Y}_1^i | W_1, \tilde{W}_1, Y_1^{i-1}, V_1^i, \tilde{V}_1^i) \Big\} \\
&\overset{(d)}{=} \sum \Big\{ I(\tilde{Y}_1^i, \tilde{V}_{1i}; Y_1^{i-1} | W_1, \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad - I(\tilde{Y}_1^i; Y_1^{i} | W_1, \tilde{W}_1, V_1^i, \tilde{V}_1^i) \\
&\qquad - I(\tilde{V}_{1i}; Y_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \Big\} \\
&\overset{(e)}{=} \sum \Big\{ I(\tilde{Y}_{1i}, \tilde{V}_{1i}; Y_1^{i-1} | W_1, \tilde{W}_1, \tilde{Y}_1^{i-1}, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad + I(\tilde{Y}_1^{i-1}; Y_1^{i-1} | W_1, \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad - I(\tilde{Y}_1^i; Y_1^{i} | W_1, \tilde{W}_1, V_1^i, \tilde{V}_1^i) \\
&\qquad - I(\tilde{V}_{1i}; Y_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \Big\} \\
&\overset{(f)}{\le} \sum \Big\{ I(\tilde{Y}_{1i}, \tilde{V}_{1i}; Y_1^{i-1} | W_1, \tilde{W}_1, \tilde{Y}_1^{i-1}, V_1^{i-1}, \tilde{V}_1^{i-1}) \\
&\qquad - I(\tilde{V}_{1i}; Y_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \Big\} \\
&\overset{(g)}{=} \sum \Big\{ I(\tilde{Y}_{1i}; Y_1^{i-1} | W_1, \tilde{W}_1, \tilde{Y}_1^{i-1}, V_1^{i-1}, \tilde{V}_1^{i}) \\
&\qquad - I(\tilde{V}_{1i}; Y_1^{i-1}; W_1, \tilde{Y}_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \Big\} \\
&\overset{(h)}{=} \sum \Big\{ I(\tilde{Y}_{1i}; Y_1^{i-1} | W_1, \tilde{W}_1, \tilde{Y}_1^{i-1}, \tilde{V}_1^{i}) \\
&\qquad - I(\tilde{V}_{1i}; W_1, \tilde{Y}_1^{i-1} | \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \Big\}
\end{aligned}
$$

where (a) follows from the definition of triple mutual information; (b) follows from the non-negativity of mutual information; (c) follows from a chain rule and the fact that V_{1i} is a function of $(W_1, \tilde{Y}_1^i)$; (d) and (e) follow from a chain rule; (f) follows from $\sum I(\tilde{Y}_1^{i-1}; Y_1^{i-1} | W_1, \tilde{W}_1, V_1^{i-1}, \tilde{V}_1^{i-1}) \le \sum I(\tilde{Y}_1^i; Y_1^i | W_1, \tilde{W}_1, V_1^i, \tilde{V}_1^i)$; and (g) follows from a chain rule and the definition of triple mutual information; (h) follows from the fact that V_1^{i-1} and $\tilde{V}_{1i}$ are a function of $(W_1, \tilde{Y}_1^{i-2})$ and $(\tilde{W}_1, Y_1^{i-1})$, respectively.