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Interference Alignment for Cellular Networks

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Abstract— **In this paper, we propose a new way of interference management for cellular networks. We develop the scheme that approaches to *interference-free* degree-of-freedom (dof) as the number K of users in each cell increases. Also we find the corresponding bandwidth scaling conditions for typical wireless channels: multi-path channels and single-path channels with propagation delay. The scheme is based on *interference alignment*. Especially for more-than-two-cell cases where there are multiple non-intended BSs, we propose a new version of interference alignment, namely *subspace interference alignment*. The idea is to align interferences into multi-dimensional subspace (instead of one dimension) for simultaneous alignments at multiple non-intended BSs. The proposed scheme requires *finite* dimensions growing linearly with K, i.e., $\sim O(K)$.**

I. INTRODUCTION

Over the past decade a great deal of research has been conducted to find or approach to the capacity of cellular networks. However, even in the simplest case (the two-user interference channel), the capacity has been open for more than 30 years. So far the only approximate capacity region was found to within one bit for all values of channel parameters [1].

In order to make progress, a simpler notion of *degree-of-freedom* (dof) has been used. With this notion, recently, Cadambe, Jafar and Shamai made significant progress on interference networks [2], [3], [4]. In [2] they showed that the dof for the K-user interference channel is $\frac{K}{2}$ assuming i.i.d. parallel channels. However, this result is not enough to apply to cellular systems since the channel itself is different. In [4], they found the dof for X-networks which are the most similar to cellular networks. The only difference is that information is allowed via cross links in X-networks while it is not in the cellular network. Hence we can interpret the cellular network as a special case of the X-network. This implies that they *implicitly* found the cellular-network dof. However, their achievable scheme has some practical problems.

First their scheme assumes i.i.d. parallel channels requiring a substantial number of multi-paths. This assumption is not realistic. The dof is still unknown under realistic channel conditions (a finite number of multi-paths or single-path). We need to evaluate how realistic channel conditions affect the dof more precisely. The second problem is that the achievable scheme requires a huge amount of dimensions that grow *exponentially* with the number of transmitter-and-receiver nodes. The scheme is based on *interference alignment*[1] of which the idea is to design transmit signals to be aligned onto one dimension at the non-intended receiver while being distinct at the desired receiver. However, applying interference alignment is not straightforward especially when there are multiple non-intended receivers. The reason is that the alignment for one receiver does not ensure alignment at the other receiver in general. To solve this problem, Cadambe and Jafar proposed a sophisticated scheme that requires a huge amount of dimensions which increase exponentially with the number of transmitter-and-receiver nodes. However, the *practical* achievable scheme that requires *finite* dimensions has been open so far.

In this paper, we show that the the *interference-free* dof can be approached as the number K of mobiles in each cell increases: for the G-cell case, the achievable dof (per cell)[2] is

$$\frac{K}{(\sqrt[G-1]{K}+1)^{G-1}} \longrightarrow 1 \quad \text{as } K \to \infty. \tag{1}$$

This is a surprising result. In the simple two-cell case where there is only one mobile in each cell (the two-user interference channel), the dof (per link) is known as $\frac{1}{2}$ if there is no collaboration of transmitters or receivers [7], [3]. We lose half of the dof due to the interference. According to our results, however, as we have more mobiles, we can approach to the *interference-free* dof, without any collaborations. This implies that *multi-user gain* can be obtained in cellular networks. We also find bandwidth scaling conditions for approaching to interference-free dof under realistic wireless channels (a finite number of multi-paths or single-path).

The major contribution of this paper is that the proposed scheme requires small dimensions of $(\sqrt[G-1]{K}+1)^{G-1}$, i.e., $\sim O(K)$. To the best of my knowledge, this is the first scheme that uses *finite* dimensions for simultaneous alignments at multiple non-intended receivers. For this scheme, we propose a new version of interference alignment, namely *subspace interference alignment*. The idea is to align interferences into multi-dimensional subspace instead of one dimension.

This paper is organized as follows. Section II describes the cellular model, wireless multi-path channels and notations.

[1]The concept was firstly introduced by Maddah-Ali *et.al* [5] and later it was well exploited by Cadambe, Jafar, and Shamai in the K-user interference channel and X-channel. It was exploited on signal *scales* as well [6].

[2]Throughout the paper, we will use the dof normalized to the number of cells.

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In Section III, for the two-cell case, we develop achievable schemes and find corresponding channel conditions. In Section IV, we propose subspace interference alignment to extend to the non-trivial three-cell case. Section V generalizes the scheme into a general number of cells and Section VI shows the uplink-downlink duality. Finally, discussion and conclusion are followed in Section VII and VIII, respectively.

II. System Model and Degree-of-Freedom

We introduce the channel model that represents cellular networks: the *interfering* multiple access channel (IMAC) in which there are multiple cells and mobiles communicate only with the desired BS. The downlink model is the *interfering* broadcast channel (IBC). Later we showed that there is the dof *duality* between the IMAC and IBC. Hence, it is enough to consider one channel only. In this paper, we focus on the IMAC.

Fig. 1 illustrates the 2 interfering multiple access channel. There are two cells of α and β, and each cell has K users. The users in each cell communicate only with the intended BS. Assume that the signal is coded over multiple orthogonal frequencies (i.e., subcarriers). The received signals at a and b are given by

$$\begin{aligned} \mathbf{y}^a &= \sum_{k=1}^{K} \mathbf{H}^a_{\alpha k}\mathbf{v}_{\alpha k}x_{\alpha k} + \sum_{k=1}^{K} \mathbf{H}^a_{\beta k}\mathbf{v}_{\beta k}x_{\beta k} + \mathbf{w}^a, \\ \mathbf{y}^b &= \sum_{k=1}^{K} \mathbf{H}^b_{\beta k}\mathbf{v}_{\beta k}x_{\beta k} + \sum_{k=1}^{K} \mathbf{H}^b_{\alpha k}\mathbf{v}_{\alpha k}x_{\alpha k} + \mathbf{w}^b, \end{aligned} \tag{2}$$

where the subscripts and superscripts denote transmitter and receiver sides, respectively and k is a user index. $\mathbf{H}^a_{\alpha k} \in \mathbb{C}^{n\times n}$, $\mathbf{v}_{\alpha k} \in \mathbb{C}^n$, $x_{\alpha k} \in \mathbb{C}$, and $\mathbf{w}^a \in \mathbb{C}^n$ indicate channels, transmit vectors, information signals and i.i.d. Gaussian noise, respectively. Here n denotes the dimension size, i.e., the number of used subcarriers. Assume that channel coefficients are known priori at transmitters and power is equally allocated to each transmitter and subcarrier.

Note that dof (per cell) is defined as

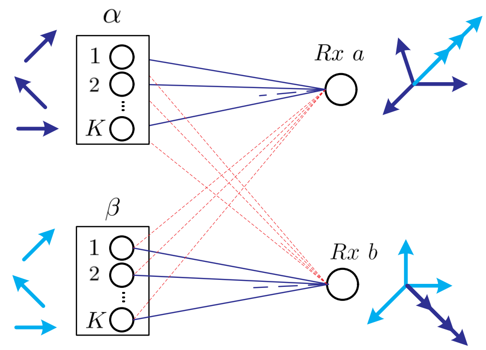
$$dof = \frac{1}{G}\lim_{\mathsf{SNR}\to\infty}\frac{C_{sum}(\mathsf{SNR})}{\log \mathsf{SNR}},$$

which indicates the pre-log value of C_{sum}. So it can be easily calculated by checking the number of linearly independent signals at the receiver. This implies that the dof is obtained with a simple zero-forcing (ZF) receiver which projects the received signal onto the space orthogonal to other signal spaces. Each BS uses a ZF vector to recover the message intended for mobile k:

$$\mathbf{u}^*_{\alpha k}\mathbf{y}^a, \mathbf{u}^*_{\beta k}\mathbf{y}^b, \quad \forall k,$$

where $(\cdot)^*$ indicates Hermitian. Note that a ZF vector $u_{\alpha k}$ depends on what the interferences are.

For the downlink model of the interfering broadcast channel (IBC), we reverse the roles of sub/superscripts, i.e., subscripts and superscripts indicate receiver and transmitter sides, respectively. Also we use tilde notations for all the



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III. 2 Interfering MAC (Two Cells)

A. Multi-Path Channels

The idea of the achievable scheme is *interference alignment* [5], [2]. In fact, the *randomness* of wireless channels is a main source for interference alignment. Hence we first consider multi-path channels having lots of randomness.

Fig. 1 illustrates the interference alignment scheme. The idea is to design transmit vectors so that those span only one dimension at the non-intended receiver. Then, due to the randomness of desired link and cross link, even if transmit signals span one dimension at the non-intended receiver, they are likely to be linearly independent at the desired receiver, i.e., they span full dimensions with high probability.

Remark: Here we can observe an interesting thing which is *multi-user gain* (many-to-one gain). Whatever K is, we reserve only one dimension for the interferences. Therefore, the efficiency becomes better as the number of mobiles increases. □

The detailed procedure is as follows. Let the dimension size $n = K+1$ and let $\mathbf{v}_r \in \mathbb{C}^{K+1}$ be a reference vector. To align interferences onto one dimension, we set $\mathbf{v}_{\alpha k}$ and $\mathbf{v}_{\beta k}$ s.t. $\forall k$,

$$\mathbf{v}_{\alpha k} = (\mathbf{H}^b_{\alpha k})^{-1}\mathbf{v}_r,\ \mathbf{v}_{\beta k} = (\mathbf{H}^a_{\beta k})^{-1}\mathbf{v}_r.$$

This setting can be done because channels are assumed to be known priori at the transmitters. Then, the received signals are

$$\begin{aligned}\mathbf{y}^a &= \sum_{k=1}^{K}\mathbf{H}^a_{\alpha k}(\mathbf{H}^b_{\alpha k})^{-1}\mathbf{v}_r x_{\alpha k} + \mathbf{v}_r\sum_{k=1}^{K}x_{\beta k} + \mathbf{w}^a,\\ \mathbf{y}^b &= \sum_{k=1}^{K}\mathbf{H}^b_{\beta k}(\mathbf{H}^a_{\beta k})^{-1}\mathbf{v}_r x_{\beta k} + \mathbf{v}_r\sum_{k=1}^{K}x_{\alpha k} + \mathbf{w}^b.\end{aligned} \tag{5}$$

Note that all the interferences are aligned along with a single reference vector $\mathbf{v}_r$, i.e., they span only one dimension.

Here the dof is determined by the rank of the matrix consisting of received vectors:

$$\begin{aligned}\mathbf{H}^a &= \left[\mathbf{H}^a_{\alpha 1}(\mathbf{H}^b_{\alpha 1})^{-1}\mathbf{v}_r, \cdots, \mathbf{H}^a_{\alpha K}(\mathbf{H}^b_{\alpha K})^{-1}\mathbf{v}_r\right],\\ \mathbf{H}^b &= \left[\mathbf{H}^b_{\beta 1}(\mathbf{H}^a_{\beta 1})^{-1}\mathbf{v}_r, \cdots, \mathbf{H}^b_{\beta K}(\mathbf{H}^a_{\beta K})^{-1}\mathbf{v}_r\right].\end{aligned} \tag{6}$$

If those have full rank and are disjoint with interference space (spanned by $\mathbf{v}_r$), then the number of distinct desired vectors is K for each cell; hence the dof per cell is achieved as $\frac{K}{K+1}$.

Now the only thing to check is the rank of $\mathbf{H}^a$ and $\mathbf{H}^b$. In fact, the rank depends on channel environment (e.g., T_d) and operation bandwidth W which determines the resolution of channel taps. Lemma 1 shows how the realistic channel conditions affect the dof.

Lemma 1 (Multi-Path Channels): Suppose W is large such that all the users have at least two significant taps, i.e., $\lceil T_{\min} W\rceil \geq 2$ where $T_{\min} = \min_k T_{d,k}$ and $T_{d,k}$ is the maximum delay spread for user k. In addition, suppose that channels have continuous distribution. Then,

$$dof = \frac{K}{K+1} \ \ a.s. \tag{7}$$

Sketch of Proof: Since each user has at least two non-zero taps, we can choose $K+1$ subcarriers out of N such that corresponding channel responses are frequency-selective for all users. Also since we assume that channels have continuous distribution, the event $\left\{\mathbf{H}^a_{\alpha k} = c\mathbf{H}^b_{\alpha k}, \forall c \in \mathbb{C}\right\}$ has zero measure, i.e., $\Pr\left(\mathbf{H}^a_{\alpha k} = c\mathbf{H}^b_{\alpha k}, \forall c \in \mathbb{C}\right) = 0, \forall k$. This implies that the space due to $\mathbf{H}^a$ is distinct with the interference space spanned by $\mathbf{v}_r$. Also since the channel is independent over different users, we have

$$\Pr\left(\mathbf{H}^a_{\alpha k} = \mathbf{H}^b_{\alpha k}\sum_{i\neq k} c_i \mathbf{H}^a_{\alpha i}(\mathbf{H}^b_{\alpha i})^{-1}\right) = 0,\ \forall c_i \in \mathbb{C}, k.$$

This guarantees the full rank of $\mathbf{H}^a$. The rigorous proof for this is omitted. We have the same results for $\mathbf{H}^b$. Therefore, we complete the proof. □

B. Single-Path Random Delay Channels

The randomness of single-path channels is much smaller than multi-path channels. So we may ask if $\mathbf{H}^a$ has full rank or not. The answer is yes as long as W goes to infinity so that channel tap-delay is pretty random.

Note that channel tap-delay corresponds to the phase shift of frequency-domain channels. So the matrix $\mathbf{H}^a$ can be written as

$$\mathbf{H}^a = \frac{1}{\sqrt{K+1}}\begin{pmatrix} W_N^{\left(\ell^a_{\alpha 1}-\ell^b_{\alpha 1}\right)\cdot 0} & \cdots & W_N^{\left(\ell^a_{\alpha K}-\ell^b_{\alpha K}\right)\cdot 0}\\ W_N^{\left(\ell^a_{\alpha 1}-\ell^b_{\alpha 1}\right)\cdot 1} & \cdots & W_N^{\left(\ell^a_{\alpha K}-\ell^b_{\alpha K}\right)\cdot 1}\\ \vdots & \ddots & \vdots\\ W_N^{\left(\ell^a_{\alpha 1}-\ell^b_{\alpha 1}\right)\cdot K} & \cdots & W_N^{\left(\ell^a_{\alpha K}-\ell^b_{\alpha K}\right)\cdot K}\end{pmatrix},$$

where $\ell^a_{\alpha k}$ denotes a tap-delay normalized to symbol rate, i.e., the real-time delay is $\frac{\ell^a_{\alpha k}}{W}$. Recall that $W_N = \exp\left(-j\frac{2\pi}{N}\right)$. For an easy interpretation, we set $v_r = \frac{\mathbf{1}}{\sqrt{K+1}} \in \mathbb{C}^{K+1}$ and $h_\ell = 1$. Clearly we see that if $\left[\ell^a_{\alpha k} - \ell^b_{\alpha k}\right]_N$ are non-zero and different for all k, then $\mathbf{H}^a$ has full rank. Here $[\cdot]_N$ is a $mod\ N$ operator. However, if W is finite, then $\ell^a_{\alpha k}$ takes a *finite integer*; hence $\Pr\left(rank(\mathbf{H}^a) = K\right) \neq 1$. The probability depends highly on W because different bandwidth provides a different number of candidate delay bins. So the achievable dof should be *statistically* calculated as a function of W and T_d.

Lemma 2 (Single-Path Channels): Assume that $\ell^j_{\alpha k}$ are i.i.d over k and $j = a, b$ with uniform distribution, i.e., $\ell^j_{\alpha k} \sim U[0, 1, \cdots, L-1]$. Then, for single-path channels,

$$dof = \frac{K}{K+1} \ \ a.s. \ \ \text{if } W \to \infty. \tag{8}$$

Moreover, $\forall \epsilon > 0$,

$$\lim_{K\to\infty} dof = 1 \ \ \text{if } W \sim O(K^{1+\epsilon}). \tag{9}$$

Proof: Let $m_k = \ell^a_{\alpha k} - \ell^b_{\alpha k}$. Then, the pmf of m_k is

$$p(m) = \frac{L-1-|m|}{(L-1)^2}, \quad -(L-1) \leq m \leq L-1. \tag{10}$$

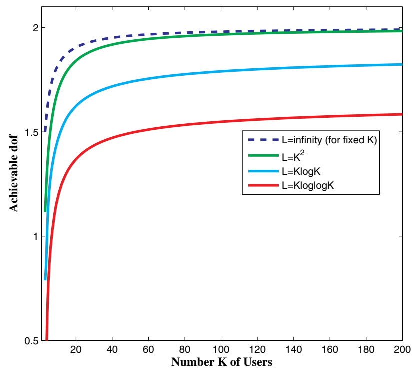


Fig. 2. The achievable degree-of-freedom for single-path random delay channels

By symmetry, we can compute the achievable dof only with the probability that user 1 signal is successfully decoded, i.e., $dof \geq \Pr(User1\ Success) \cdot \frac{K}{K+1}$. Now focus on:

$$\begin{aligned}
&\Pr\left(User1\ Success\right)\\
&= \Pr\left(m_1 \neq 0\right) \prod_{k=2}^{K} \Pr\left(m_1 \neq m_k | m_1 \neq 0, m_1 \neq m_i \forall i \leq k-1\right)\\
&= \Pr\left(m_1 \neq 0\right) \left[\Pr\left(m_1 \neq m_2 | m_1 \neq 0\right)\right]^{K-1}\\
&= \frac{L-2}{L-1} \cdot \left(1 - \frac{2L-3}{3(L-1)^2}\right)^{K-1},
\end{aligned}$$

where the third equation is because the events $\{m_1 \neq m_i\}$ and $\{m_1 \neq m_j\}$ are independent for $i \neq j$. Letting $L \to \infty$, we prove (8).

For the second proof, let us rewrite the second term as $[(1 - \frac{2L-3}{3(L-1)^2})^L]^{\frac{K-1}{L}}$. Letting L go to infinity for fixed K, the inside term converges to $\exp\left(-\frac{2K}{3L}\right)$. So if $\frac{K}{L} \to 0$ and $K \to \infty$, we get the interference-free dof. This proves (9). ■

Lemma 2 says that the condition $W = O(K \log K)$ can be sufficient to achieve the interference-free dof. However, we may want much stricter condition to get faster convergence rate, since the convergence rate is important in practical systems due to the limited K. Fig. 2 shows how convergence rate behaves for different order of L. We can see that $L = O(K^2)$ is an appropriate order for obtaining efficient dof.

IV. 3 Interfering MAC (Three Cells)

From the three-cell case, the achievable scheme used in [4] requires a huge amount of dimensions which exponentially grow with the number of transmitter-and-receiver nodes. This asks for an impractical bandwidth size. In this section, we propose a new version of interference alignment, called *subspace interference alignment*, to develop the practical scheme requiring finite dimensions $\sim O(K)$. Also we find the bandwidth scaling condition to approach to the interference-free dof. This section is the main contribution of our paper.

A. Subspace Interference Alignment

Aligning interferences becomes challenging from the three-cell case because there are multiple non-intended receivers (interferees). Interference alignment for one receiver does not ensure the alignment in the other receiver. So the simple interference alignment employed in 2-IMAC cannot be applied to this case. In fact, the problem is mainly because we align interferences into only *one dimension*. This constraint seems so strict. We solve this problem by relaxing the constraint. The idea is to align interferences into *multi-dimensional subspace* instead of one dimension.

Goal: Fig. 3 illustrates the conceptual *goal* of subspace interference alignment for 3-IMAC. Later we will explain how to accomplish the goal in realistic wireless channels. We have difficulty in explaining the main idea because it is hard to visualize subspace. So we abstract subspace with simple grids. For example, to represent a vector spanning one dimension, we use one grid. For the two-dimensional plane, we use two grids. For the three-dimensional subspace, we use three grids.

Suppose that the whole space is decomposed into two subspaces so that the dimension of the whole space is the product of the dimension of each subspace. Similarly assume that a transmit vector is decomposed into two subspace vectors. The idea is to design subspace 1 vectors for subspace 1 alignment at one receiver; at the same time, to design subspace 2 vectors for subspace 2 alignment at the other receiver. For example, in cell α, we design subspace 1 vectors so that those signals span only one dimension of subspace 1 (subspace 1 alignment) but span multiple dimensions of subspace 2 at receiver b. At the same time we design subspace 2 for subspace 2 alignment at receiver c. At the desired receiver a, any spaces are not aligned so the signals are distinct. Now suppose each user transmits only one symbol. Then, we need K dimensions for desired signals. To get this value, we set the dimension of each subspace as $\sqrt{K}+1$, resulting in $(\sqrt{K}+1)^2$ dimensions for the whole space. Therefore, under this setting, we can expect the following dof (per cell):

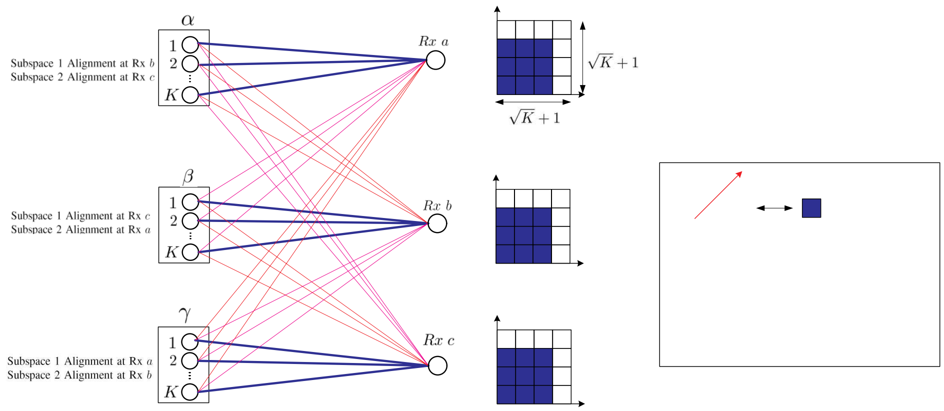
$$\frac{K}{(\sqrt{K}+1)^2}. \tag{11}$$

The main idea of the *subspace interference alignment* is to align K interfering vectors into $\sqrt{K}+1$ dimensions (instead of one dimension) to enable simultaneous alignments at multiple interferee receivers. Since $\sqrt{K}$ becomes negligible compared to K as K gets large, we can approach to the interference-free dof.

Scheme: Now the question is how to accomplish the goal in realistic wireless channels. To obtain the goal, we exploit the special property of a special type of channels, which is namely *decomposability*.

Definition 1: Suppose we can decompose $\mathbf{H} \in \mathbb{C}^{n \times n}$ as

$$\mathbf{H} = \bigotimes_{i=1}^{m} \mathbf{H}^i := \mathbf{H}^m \otimes \mathbf{H}^{m-1} \otimes \cdots \otimes \mathbf{H}^1.$$



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collapsed into $\sqrt{K}+1$ since it has a *degenerated* structure. Note that the left-side term is fixed as $\mathbf{v}_r$ for all the users. The randomness comes only from the right-side vector $\mathbf{H}^{a,1}_{\beta k}(\mathbf{H}^{c,1}_{\beta k})^{-1}\mathbf{v}_r$ which has dimension $\sqrt{K}+1$. Therefore, the rank is limited by $\sqrt{K}+1$. Similarly, the interference space from γ has the rank of at most $\sqrt{K}+1$.

Now check the rank of desired signal space. By $(C1)$, the rank of desired signal space is K. The condition $(C2)$ guarantees that the desired signal space are *disjoint* with interference spaces from β and γ. This implies that we can recover K messages from $\mathbf{y}^a$. Similarly, we do the same procedures for $\mathbf{y}^b$ and $\mathbf{y}^c$ to obtain $3K$ messages in total. Therefore, we obtain (11). ∎

This theorem gives insights into finding the practical scheme for more-than-two-cell cases. As the theorem says, the key issue is to find *decomposable* channels from real wireless channels. One good thing is that channels are not limited to a diagonal structure. Even for a non-diagonal matrix (e.g., multi-antenna channels), subspace interference alignment works as long as $\mathbf{H}$ is *decomposable*. One application of the theorem can be found in single-path random delay channels.

B. Single-Path Random Delay Channels

Proposition 1: Single-path channels are *2-level decomposable*.

Proof: The frequency response of signal-path channels is given by $H[f]=h_\ell W_N^{\ell f}$. Write f in terms of two frequencies f_1 (local frequencies) and f_2 (global frequencies):

$$f=(\sqrt{K}+1)f_2+f_1,\ \forall\ f_1,f_2\in\left\{0,1,\cdots,\sqrt{K}\right\},$$

Then we can decompose the channel as:

$$H[f]=\left(h_\ell W_N^{\ell(\sqrt{K}+1)f_2}\right)\cdot\left(W_N^{\ell f_1}\right). \tag{14}$$

Letting $H^1[f_1]=W_N^{\ell f_1}$ and $H^2[f_2]=h_\ell W_N^{\ell(\sqrt{K}+1)f_2}$, we get

$$\begin{aligned}\mathbf{H}&=diag\left(H[f]\right)_f=diag\left(H^2[f_2]H^1[f_1]\right)_{f_1,f_2}\\&=diag\left(H^2[f_2]\right)_{f_2}\otimes diag\left(H^1[f_1]\right)_{f_1}\\&=\mathbf{H}^2\otimes\mathbf{H}^1.\end{aligned} \tag{15}$$

This completes the proof. ∎

Lemma 3 tells how bandwidth scaling conditions affect the dof depending on $(C1)$ and $(C2)$ in (13).

Lemma 3: For single-path random delay channels,

$$dof\geq\max\left(\frac{2}{3}\cdot\frac{K}{K+1},\frac{K}{(\sqrt{K}+1)^2}\right)\quad\text{if }\ W\to\infty. \tag{16}$$

Moreover, $\forall\epsilon>0$,

$$\lim_{K\to\infty}dof=1\quad\text{if }\ W\sim O(K^{1+\epsilon}). \tag{17}$$

Proof: See Appendix I. ∎

Notice that the condition for approaching to the interference-free dof is the same as the two-cell case.

C. Extension to Multi-Path Channels

One may ask if multi-path frequency-selective channels are *decomposable* as well. Unfortunately, the answer is no in general. However, we can *indirectly* apply subspace interference alignment. The idea is to chop up the whole band into sub-bands within coherence bandwidth. By Proposition 1, then, the channel is decomposable within the sub-band. This can be seen from the following simple example. Suppose that the channel has two non-zero taps at ℓ_1 and ℓ_2.

$$H[f]=h_{\ell_1}W_N^{\ell_1 f}+h_{\ell_2}W_N^{\ell_2 f}=W_N^{\ell_1 f}\left(h_{\ell_1}+h_{\ell_2}W_N^{(\ell_2-\ell_1)f}\right).$$

Since the coherence bandwidth is $W_c\triangleq\frac{W}{2(\ell_2-\ell_1)}$, the term $(h_{\ell_1}+h_{\ell_2}W_N^{(\ell_2-\ell_1)f})$ is almost constant within W_c. This implies that the channel virtually has a single tap within sub-band. Hence, we can apply subspace interference alignment for each sub-band.

For highly frequency selective channels, however, we may have some practical problems because the number of subcarriers within coherence bandwidth might be so small that we lose efficiency of dof a lot. But this problem can be resolved by making subcarrier spacing small enough to have many subcarriers within sub-band. However, small subcarrier spacing might cause inter-carrier interference (due to Doppler effect) and increase hardware complexity (due to large N). Therefore, given bandwidth W, the IDFT/DFT size N should be appropriately chosen considering all of these effects. The achievable scheme that is *directly* applicable to frequency-selective channels is open.

V. G INTERFERING MAC (G CELLS)

We can easily extend the subspace interference alignment into a general number of cells. For explain this, we use *number* notations instead of symbol ones. For example, H^1_{0k} denotes channel from user k in cell 0 to receiver 1. We start with number 0 for cell index because of a modular operation.

Theorem 2 (Generalized Subspace Interference Alignment): Let $n=(\sqrt[G-1]{K}+1)^{G-1}$. Suppose that $\forall i,j\in\{0,1,\cdots,G-1\}$,

$$\mathbf{H}^j_{ik}=\bigotimes_{g=0}^{G-2}\mathbf{H}^{j,g}_{ik}=\mathbf{H}^{j,G-2}_{ik}\otimes\mathbf{H}^{j,G-3}_{ik}\otimes\cdots\otimes\mathbf{H}^{j,0}_{ik}. \tag{18}$$

Let $\forall j=1,2,\cdots,G-1$,

$$\begin{aligned}\mathbf{H}^j=&\left[\bigotimes_{g=0}^{G-2}\mathbf{H}^{j,g}_{j1}(\mathbf{H}^{[j+g+1]_G,g}_{j1})^{-1}\mathbf{v}_r,\cdots,\right.\\&\left.\bigotimes_{g=0}^{G-2}\mathbf{H}^{j,g}_{jK}(\mathbf{H}^{[j+g+1]_G,g}_{jK})^{-1}\mathbf{v}_r\right],\end{aligned} \tag{19}$$

where $\mathbf{v}_r\in\mathbb{C}^{\sqrt[G-1]{n}}$. Suppose that following two conditions hold:

$$\begin{aligned}&(C1):rank(\mathbf{H}^i)=K,\quad\forall i\in\{0,1,\cdots,G-1\},\\&(C2):\mathbf{H}^{i,g}_{ik}\neq\mathbf{H}^{[i+g+1]_G,g}_{ik},\ \forall i,g\in\{0,1,\cdots,G-1\},\\&\qquad\qquad\forall k\in\{1,2,\cdots,K\}.\end{aligned} \tag{20}$$

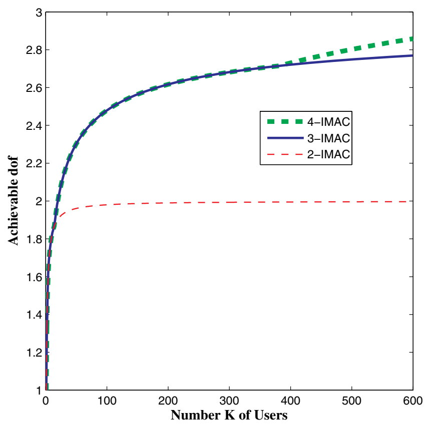


Fig. 4. The achievable degree-of-freedom for the proposed scheme

Then, there exists the scheme that achieves

$$\frac{K}{(\sqrt[G-1]{K}+1)^{G-1}}. \tag{21}$$

Proof: The idea is essentially same as the proof of Theorem 1. So we omit the proof. ■

We can easily verify that single-path channels are $(G-1)$-level decomposable. Hence, similar to the three-cell case, we have the following lemma.

Lemma 4: For single-path random delay channels,

$$dof \geq \max_{2\leq g\leq G} \frac{g}{G}\cdot\frac{K}{(\sqrt[g-1]{K}+1)^{g-1}} \quad \text{if } W\to\infty. \tag{22}$$

Moreover, $\forall\epsilon>0$,

$$\lim_{K\to\infty} dof = 1 \quad \text{if } W\sim O(K^{1+\epsilon}). \tag{23}$$

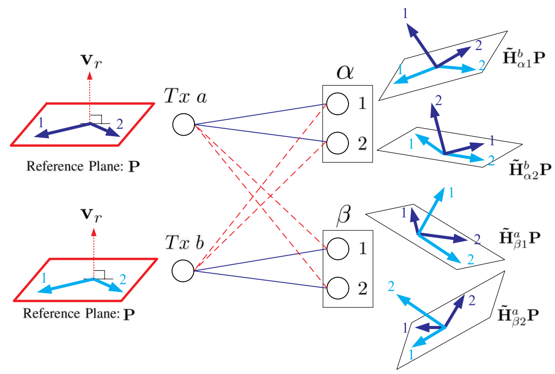
Proof: The idea is similar to the proof of Lemma 3. So we omit it. ■

Similar to the three-cell case, we can also extend into multi-path channels by chopping up the whole band into sub-band within coherence bandwidth. Therefore, Lemma 4 still holds for the frequency-selective multi-path channel.

However, we have limitations in applying into practical systems. Note that very large K is required to obtain high efficiency of the achievable dof. The larger G, the worse the efficiency. Fig. 4 shows that the achievable dof has bad efficiency especially from the four-cell case. However, this problem is not significant because there are three neighboring BSs in typical hexagonal cellular systems; hence the four-cell case is not interesting. In fact, the achievable scheme for the three-cell case is most appropriate for a practical range of $K(\leq 50)$.

VI. Uplink-Downlink Duality

The interference alignment scheme for the interference broadcast channel (IBC) is not as simple as that for the IMAC, since there are many non-intended receivers. However, an easy interpretation can be made especially for 2-IBC by using a new concept: interference alignment on the *plane* instead of a vector.



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Here we can see interesting relationship between IMAC and IBC. Note that the roles of a reference vector, transmit vectors, and ZF vectors in the IMAC correspond to the roles of a reference *plane*, ZF vectors, and transmit vectors in the IBC. It turns out that there is the dof *duality* between IMAC and IBC.

To cover $G(\geq 3)$-IBC, we use the smart *duality* property, which holds for many kinds of channels: MAC/BC [8]; many-to-one/one-to-many interference channels [6]; (M-by-N)/(N-by-M) X-networks [4].

Theorem 3: There is the dof duality between G-IMAC and G-IBC, i.e., if there exists the achievable scheme for one channel, so does there for the other channel.

Proof: The proof is similar to the X-network duality proof of which the idea is to use a ZF receiver architecture. So we omit it. ■

VII. Discussion

The proposed scheme well exploited the concept of interference alignment to approach to *interference-free* dof in cellular networks. In fact, the benefit of the interference alignment comes from *many-to-one gain (multi-user gain)*. Hence, the proposed scheme is very much suitable for cellular networks which is the best well-known example of many-to-one/one-to-many links. Due to this property, it does not have gains for the K interfering point-to-point link (the K-user interference channel).

The proposed scheme looks promising because it enables *interference-free* cellular communications. However, it has still practical issues. First, we need to know how the proposed scheme works for all range of SNR, since the practical range of SNR is $5 \sim 20$ dB. The next practical problem is that channel information should be known at the transmitters beforehand. Since the scheme requires full channel information (both magnitude and phase), the overhead could be significant. In order to tackle this problem, we may think of a *distributed* interference alignment scheme which has iterative procedures and requires only partial channel state information at each iteration.

Finally, we would like to emphasize the potential of *subspace interference alignment* which played a crucial role in generalizing the achievable scheme. Notice that the theorem 1, 2 are not limited to diagonal matrices. The channel matrix for multi-antenna systems has a non-diagonal structure. So we have potential to extend into MIMO cellular networks by finding the *decomposable* matrices from MIMO channels.

VIII. Conclusion

In this paper, we considered an *interfering* multiple access channel for cellular networks (a subnetwork of X-networks). We exploited many-to-one gain (multi-user gain) of interference alignment in cellular networks. Also we exploited a special structure of single-path channels: *decomposability*. Based on these, we developed a new interference alignment scheme that approaches to the interference-free dof; and found the corresponding bandwidth scaling conditions, i.e., $\sim O(K^{1+\epsilon})$. Especially for more-than-two-cell cases ($G \geq 3$), we proposed a new type of interference alignment, called *subspace interference alignment*, which is to align interferences into multi-dimensional subspace instead of one dimension. The proposed scheme requires *finite* dimensions which grow linearly with K, i.e., $\sim O(K)$. This is the first scheme that uses finite dimensions for aligning interferences simultaneously when there are multiple non-intended receivers.

Appendix I
Proof of Lemma 3

$dof \geq \frac{2}{3} \cdot \frac{K}{K+1}$ is straightforward by Lemma 2. The remaining proof is also very similar to Lemma 2. The only difference is that the delay collision occurs in a more complicated manner. For simplicity, we set $v_r = \frac{\mathbf{1}}{\sqrt{K+1}} \in \mathbb{C}^{\sqrt{K}+1}$ and $h_\ell = 1$. Let $m_k^b = \ell_{\alpha k}^a - \ell_{\alpha k}^b$ and $m_k^c = \ell_{\alpha k}^a - \ell_{\alpha k}^c$. Assume that $\ell_{\alpha k}^j$ are i.i.d. uniform over k and $j = a, b, c$. Then, m_k^b and m_k^c have the pmf (10). By symmetry, it is enough to consider the probability that user 1 signal is successfully decoded. For simplicity, make N sufficiently large so that $L(\sqrt{K}+1) < N$. Let $n = (\sqrt{K}+1)^2$. Compute:

$$
\begin{aligned}
&\Pr(User1\ Success) = \Pr\left(m_1^c\sqrt{n} + m_1^b \neq 0\right) \\
&\cdot \prod_{k=2}^{K} \Pr\left((m_1^c - m_k^c)\sqrt{n} + m_1^b - m_k^b \neq 0 | m_1^c\sqrt{n} + m_1^b \neq 0,\right. \\
&\qquad \left.(m_1^c - m_i^c)\sqrt{n} + m_1^b - m_i^b \neq 0, \forall i \leq k-1\right) \\
&\geq \Pr\left(m_1^b \neq 0, m_1^c \neq 0\right) \\
&\cdot \left[\Pr\left(m_2^c \neq m_1^c, m_2^b \neq m_1^b | m_1^b \neq 0, m_1^c \neq 0\right)\right]^{K-1} \\
&\simeq \left(\frac{L-2}{L-1}\right)^2 \cdot \left(1 - \frac{2L-3}{3(L-1)^2}\right)^{2K-2},
\end{aligned}
$$

where we used the lower bound technique in the second line and assumed m_k^b and m_k^c are independent (not true but close to independence) in the third line for simplifying the calculation. This completes the proof.

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