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Degrees of Freedom of the MIMO Rank-deficient Interference Channel with Feedback

Sung Ho Chae, Changho Suh, and Sae-Young Chung
Department of Electrical Engineering
KAIST, Daejeon, South Korea
shchae@kaist.ac.kr, chsuh@kaist.ac.kr, and sychung@ee.kaist.ac.kr

Abstract— **We investigate the total degrees of freedom (DoF) of the K-user rank-deficient interference channel with feedback. We focus on a symmetric case in which each node has the same number of antennas, the rank of each direct link is the same, and the rank of each cross link is the same. In this paper, we develop a new achievable scheme which employs interference alignment to efficiently utilize the dimension of the received signal space. In addition, we derive an upper bound for the general K-user case and show the tightness of the bound when the number of antennas at each node is sufficiently large. As a consequence of this result, we also show that if there are enough number of antennas at each node, the DoF gain increases proportionally with the number of users.**

I. INTRODUCTION

Recently, it has been shown that feedback can provide a significant capacity gain in the Gaussian interference channel [1]. In contrast to the point-to-point channel [2] and the multiple access channel [3], in the interference channel the gain due to feedback becomes arbitrarily large for certain channel parameters (unbounded gain). The gain comes from the fact that feedback can help efficient resource sharing between the interfering users.

The results of [1] indicate that feedback enables a significant capacity improvement of multi-user networks with interfering links. However, if we turn our attention to degrees of freedom (DoF), feedback fails to provide promising results. From the results of [4], [5], it has been shown that feedback cannot improve the total DoF for the two-user full-rank Gaussian MIMO interference channel[1]. Therefore, feedback can provide unbounded capacity gain but cannot increase the DoF in the full-rank channel.

However, recently in [7], we have shown that feedback can increase the total DoF in the two-user *rank-deficient* interference channel. Here, the rank-deficient channel captures a poor scattering environment where there are only few signal paths between nodes. In [7], we have characterized the total DoF by developing an achievable scheme and deriving a matching upper bound. In addition, as a consequence of this result, we show that feedback can increase the DoF when the number of antennas at each node is large enough as compared to the ranks of channel matrices. This finding is in contrast to the full-rank interference channel where feedback provides no DoF gain. The key idea is that feedback can provide alternative signal paths, thereby effectively increasing the ranks of desired channel matrices.

In this paper, we extend the previous works in [7] to the general K-user case. We focus on a symmetric case in which each node has the same number of antennas, the rank of each direct link is the same, and the rank of each cross link is the same. Specifically, for $K = 3$, we establish the achievable total DoF by developing a new achievable scheme. As compared to the two-user case, a significant distinction of the proposed scheme is that we can now employ interference alignment to efficiently utilize the dimension of the received signal space, especially when the rank of cross links is sufficiently large as compared to the number of antennas at each node. In addition, we derive an upper bound for the general K-user case and show the tightness of the bound when the number of antennas at each node is sufficiently large. One interesting consequence of this result is that if we can use sufficiently many antennas at each node, the DoF gain increases proportionally with the number of users.

Notations: Throughout the paper, we will use $\mathbf{A}$ and $\mathbf{a}$ to denote a matrix and a vector, respectively. Let $\mathbf{A}^T$ and $||\mathbf{A}||$ denote the transpose and the norm of $\mathbf{A}$, respectively. In addition, let $|\mathbf{A}|$ and $\text{rank}(\mathbf{A})$ denote the determinant and the rank of $\mathbf{A}$, respectively. The notations $\mathbf{I}_n$ and $\mathbf{0}_{n\times n}$ denote the $n \times n$ identity matrix and zero matrix, respectively. We write $f(x) = o(x)$ if $\lim_{x\to\infty} \frac{f(x)}{x} = 0$. For convenience, when indexing channel matrices, we use modular arithmetic where the modulus is the number of users. (e.g., for the three-user case, $\mathbf{H}_{1,4}$ means $\mathbf{H}_{1,1}$).

II. SYSTEM MODEL

Consider the K-user rank-deficient interference channel with feedback, as depicted in Fig. 1. Transmitter i wishes to communicate with receiver i, and transmitter i and receiver i use M_i and N_i antennas, respectively. We assume that all channel coefficients are fixed and known to all nodes. Then, the input and output relationship at time slot t is given by

$$\mathbf{y}_j(t) = \sum_{i=1}^{K} \mathbf{H}_{j,i}\mathbf{x}_i(t) + \mathbf{z}_j(t),$$

where $\mathbf{x}_i(t)$ is the $M_i \times 1$ input signal vector at transmitter i, $\mathbf{H}_{j,i}$ is the $N_j \times M_i$ channel matrix from transmitter i to

[1]However, recently it has been shown in [6] that for *multihop* networks, feedback can increase DoF.

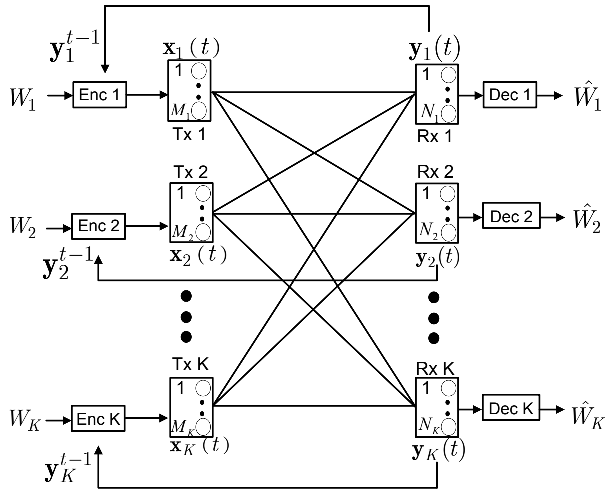

Fig. 1. The K-user rank-deficient interference channel with feedback.

receiver j, and $\mathbf{y}_j(t)$ is the $N_j \times 1$ received signal vector at receiver j. The noise vector $\mathbf{z}_j(t)$ is the additive white circularly symmetric complex Gaussian with zero mean and covariance of $\mathbf{I}_{N_j}$. We assume that all of the noise vectors and signal vectors are independent of each other.

In this paper, we adopt the rank-deficient channel model in [8], in which there are $D_{j,i} \le \min\{M_i, N_j\}$ independent signal paths from transmitter i to receiver j. Let $\mathbf{H}_{j,i}^{(k)}$ denote the channel matrix corresponding to the kth signal path between transmitter i and receiver j. Note that due to the key-hole effect [8], rank$(\mathbf{H}_{j,i}^{(k)}) = 1$, $\forall k = 1, 2, \ldots, D_{j,i}$. Therefore, we assume that the matrix $\mathbf{H}_{j,i}$ is given by

$$\begin{aligned}\mathbf{H}_{j,i} &= \sum_{k=1}^{D_{j,i}} \mathbf{H}_{j,i}^{(k)} \\ &= \sum_{k=1}^{D_{j,i}} \mathbf{a}_{j,i}^{(k)} \mathbf{b}_{j,i}^{(k)T}, \quad \forall i, j = 1, 2, \ldots, K \qquad (1)\end{aligned}$$

where $\mathbf{a}_{j,i}^{(k)}$ and $\mathbf{b}_{j,i}^{(k)}$ are $N_j \times 1$ and $M_i \times 1$ vectors respectively, and their coefficients are drawn from a continuous distribution. From (1), we can see that rank$(\mathbf{H}_{j,i}) = D_{j,i}$ with probability one.

There are K independent messages $W_1, W_2, \ldots, W_K$. At time slot t, transmitter i sends the encoded signal $\mathbf{x}_i(t)$, which is a function of W_i and past output sequences $\mathbf{y}_i^{t-1} \triangleq [\ \mathbf{y}_i(1) \quad \mathbf{y}_i(2) \quad \cdots \quad \mathbf{y}_i(t-1)\ ]^T$. We assume that each transmitter should satisfy the average power constraint P, i.e., $E[|\mathbf{x}_i(t)|^2] \le P$ for $i \in \{1, 2, \ldots, K\}$. A rate tuple $(R_1, R_2, \ldots, R_K)$ is said to be achievable if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, \ldots, , 2^{nR_K}, n)$ codes such that the average probability of decoding error tends to zero as the code length n goes to infinity. The capacity region $\mathcal{C}$ of this channel is the closure of the set of achievable rate tuples $(R_1, R_2, \ldots, R_K)$. The total DoF is defined as $\Gamma = \lim_{P\to\infty} \max_{(R_1, R_2, \ldots, R_K)\in\mathcal{C}} \frac{\sum_{i=1}^K R_i}{\log(P)}$.

III. Main Results

We first review the previous results for the two-user case in [7].

Theorem 1 (Two-user case [7]): For the two-user rank-deficient interference channel with feedback, the total DoF is given by

$$\begin{aligned}\Gamma_{fb} = \min\{&M_1 + N_2 - D_{2,1}, M_2 + N_1 - D_{1,2}, \\ &D_{1,1} + D_{2,2} + D_{1,2}, D_{1,1} + D_{2,2} + D_{2,1}, \\ &\min\{M_1, N_1\} + D_{2,2}, \min\{M_2, N_2\} + D_{1,1}\}\end{aligned}$$

Proof: See [7] for the proof. ■

Remark 1: The DoF gain due to feedback can be achieved when channel matrices of desired links are highly rank-deficient. In this case, feedback can provide alternative signal paths, thereby effectively increasing the ranks of desired channel matrices.

Now we explain the main results of this paper. When $K \ge 3$, we focus on a *symmetric* case where $M_i = N_i = M$, $D_{i,i} = D_d$, and $D_{j,i} = D_c$, $\forall i = 1, 2, \ldots, K$ and $i \ne j$. Specifically, for $K = 3$, we develop a new achievable scheme which employs interference alignment when the rank of cross links D_c is sufficiently large. The achievable total DoF for the three-user case is stated in the following theorem.

Theorem 2 (Lower bound for $K = 3$): For the symmetric three-user rank-deficient interference channel with feedback, the following total DoF is achievable.

$$\Gamma_{fb} \ge \begin{cases} M + D_d & \text{if } 2D_d \le M \le \min\{2D_c, D_d + D_c\}, \\ 2M - D_c & \text{if } \max\{2D_d, D_d + D_c\} \le M \le 2D_c\ , \\ \frac{3M}{2} & \text{if } M \le \min\{2D_c, 2D_d\}\ , \\ 3M - 3D_c & \text{if } 2D_c \le M \le 2D_c + D_d, \\ 3D_d + 3D_c & \text{if } 2D_c + D_d \le M. \end{cases}$$

Proof: See Section IV for the proof. ■

Remark 2: If there is no feedback, the total DoF is [9]:

$$\Gamma_{no} = \begin{cases} 3D_d & \text{if } 2D_d \le M \le \min\{2D_c, D_d + D_c\}, \\ 3D_d & \text{if } \max\{2D_d, D_d + D_c\} \le M \le 2D_c, \\ \frac{3M}{2} & \text{if } M \le \min\{2D_c, 2D_d\}, \\ 3M - 3D_d & \text{if } 2D_c \le M \le D_c + D_d, \\ 3D_d & \text{if } \max\{2D_c, D_c + D_d\} \le M. \end{cases}$$

Therefore, we can alternatively express Theorem 2 as

$$\Gamma_{fb} \ge \begin{cases} \Gamma_{no} & \\ +M - 2D_d & \text{if } 2D_d \le M \le \min\{2D_c, D_d + D_c\}, \\ \Gamma_{no} + 2M & \\ -D_c - 3D_d & \text{if } \max\{2D_d, D_d + D_c\} \le M \le 2D_c, \\ \Gamma_{no} & \text{if } M \le \min\{2D_c, 2D_d\}\ , \\ \Gamma_{no} & \text{if } 2D_c \le M \le D_c + D_d, \\ \Gamma_{no} + 3M- & \\ 3D_c - 3D_d & \text{if } \max\{2D_c, D_d + D_d\} \le M \le 2D_c + D_d, \\ \Gamma_{no} + 3D_c & \text{if } 2D_c + D_d \le M. \end{cases}$$

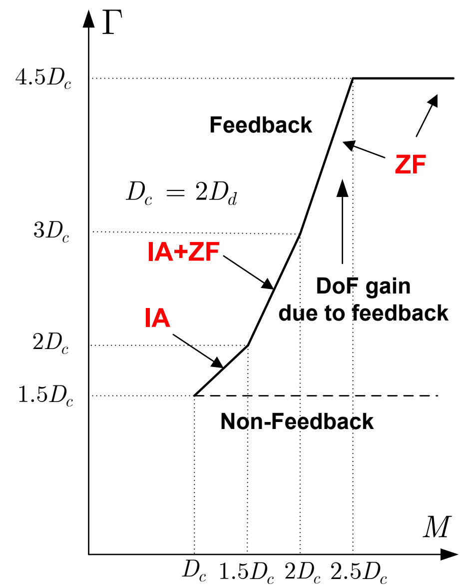

Fig. 2. Achievable total DoF for the three-user case when $D_c = 2D_d$. The achievable scheme is based on zero-forcing (ZF) and/or interference alignment (IA), depending on the number of antennas at each node.

Remark 3: As will be explained in Section IV, our achievable scheme involves interference alignment with feedback when $D_c \leq M < 2D_c$, while it is merely based on zero-forcing when $M \geq 2D_c$. This is due to the fact that when the ratio of D_c to M is greater than a certain threshold, we cannot null out all the interference signals, and thus aligning unwanted signals is required to utilize the dimension of the received signal space more efficiently. Furthermore, as will be shown in Theorem 3, the proposed scheme achieves the optimal total DoF when $M \geq 2D_c + D_d$.

DoF gain due to feedback: Consider the case where $D_c = 2D_d$. We plot the total DoF as a function of M with fixed D_c and D_d in Fig. 2. Note that we employ interference alignment when the rank of each cross link D_c is sufficiently large as compared to the number of antennas at each node M (Here, when $M \leq 2D_c$). In addition, we can see that the slope in Fig. 2 increases with the number of antennas. This is because if there are enough antennas at each node, we can even create new interference-free signal paths via zero-forcing rather than aligning unwanted signals.

We provide a simple example that shows how interference alignment can be applied when feedback is present.

Example 1: Consider the case where $M = D_c = 5$ and $D_d = 1$. Our achievable scheme operates in two time slots. See Fig. 3. At time slot 1, we design the transmitted signal for transmitter $i \in \{1, 2, 3\}$ as

$$\mathbf{x}_i(1) = \mathbf{v}_i^{[1]} s_{i,1} + \mathbf{v}_i^{[2]} s_{i,2} + \mathbf{v}_i^{[3]} s_{i,3}.$$

Here, transmitter i delivers $s_{i,1}$, $s_{i,2}$, and $s_{i,3}$ to receivers i, $i+1$, and $i+2$, respectively, while aligning unwanted signals for each receiver. Note that although $s_{i,2}$ and $s_{i,3}$ are not intended symbols for receivers $i+1$ and $i+2$, using

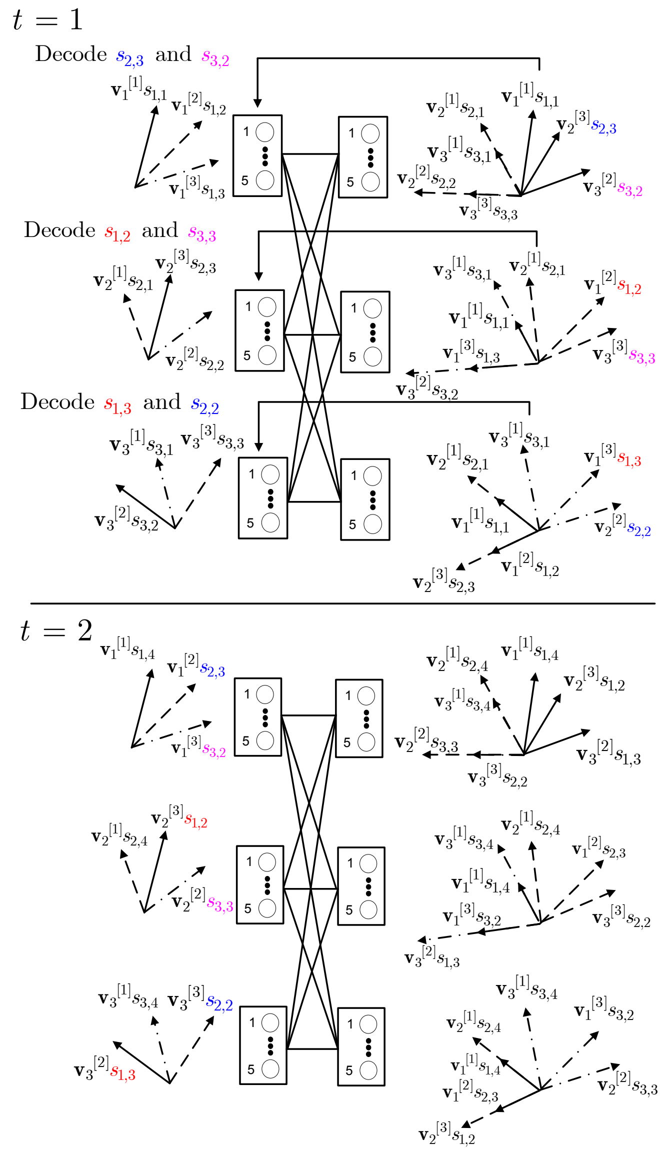

Fig. 3. Achievability in Example 1. The beamforming vectors represented by solid, dashed, and dashed-dotted lines denote the desired signals for receivers 1, 2, and 3 respectively at each time slot. Note that arrows in the figure represent linearly independent directions in a five dimensional space.

feedback, transmitters $i+1$ and $i+2$ will forward them to receiver 1 in the next time slot.

To achieve these, we construct $\mathbf{v}_i^{[1]}$ such that

$$\text{span}\left(\mathbf{H}_{i+1,i}\mathbf{v}_i^{[1]}\right) \subseteq \text{span}\left(\mathbf{H}_{i+1,i+2}\mathbf{v}_{i+2}^{[1]}\right).$$

We also design $\mathbf{v}_i^{[2]}$ and $\mathbf{v}_i^{[3]}$ such that

$$\begin{aligned}\mathbf{H}_{i,i}\mathbf{v}_i^{[3]} &= \mathbf{H}_{i+2,i+2}\mathbf{v}_{i+2}^{[2]} = 0,\\ \mathbf{H}_{i+1,i}\mathbf{v}_i^{[3]} &= \mathbf{H}_{i+1,i+2}\mathbf{v}_{i+2}^{[2]}.\end{aligned}$$

This beamforming design is feasible for $M \geq 2D_d$ and $M \leq 2D_c$ (This will be clarified in Section IV.). It turns out that for receiver $i+1$, unwanted symbols ($s_{i,1}$, $s_{i+2,1}$) and ($s_{i,3}$,

$s_{i+2,2}$) are aligned. Now we can see that

$$\text{rank}\left(\left[\begin{array}{cc}\mathbf{H}_{i,i+1}\mathbf{v}_{i+1}^{[1]} & \mathbf{H}_{i,i+2}\mathbf{v}_{i+2}^{[1]}\end{array}\right]\right)=1,$$
$$\text{rank}\left(\left[\begin{array}{cc}\mathbf{H}_{i,i+1}\mathbf{v}_{i+1}^{[2]} & \mathbf{H}_{i,i+2}\mathbf{v}_{i+2}^{[3]}\end{array}\right]\right)=1$$

with probability one. Hence, receiver i can decode $s_{i,1}$.

On the other hand, transmitter i can know $s_{i+2,2}$ and $s_{i+1,3}$ after receiving $\mathbf{y}_i(1)$. At the next time slot, each transmitter forwards the other user's symbols in addition to its own fresh symbol. To achieve this, we design the transmitted signal as

$$\mathbf{x}_i(2)=\mathbf{v}_i^{[1]}s_{i,4}+\mathbf{v}_i^{[2]}s_{i+1,3}+\mathbf{v}_i^{[3]}s_{i+2,2},$$

where $s_{i,4}$ is a new symbol for user i. Then, using the same argument above, we can see that receiver i can decode $(s_{i,2}, s_{i,3}, s_{i,4})$, $\forall i = 1, 2, 3$. As a result, we can send 12 symbols over two time slots, thus achieving $\Gamma_{fb} \geq 6$. Note that the total DoF becomes three when there is no feedback.

A. Upper bound for the K-user case

Theorem 3 (Upper bound for the K-user case): For the symmetric K-user rank-deficient interference channel with feedback, the total DoF is upper bounded by

$$\Gamma_{fb}\leq KD_d+\frac{D_cK(K-1)}{2}.$$

Proof: See Section V for the proof. ■

Corollary 1: For the symmetric K-user rank-deficient interference channel with feedback, the total DoF is given by

$$\Gamma_{fb}=KD_d+\frac{D_cK(K-1)}{2}$$

when $M \geq D_d + (K-1)D_c$.

Proof: The converse follows from Theorem 3. For achievability, we consider a simple extension of the scheme in Theorem 2. At the first time slot, each transmitter sends total $D_d + (K-1)D_c$ symbols, in which D_d symbols are sent through the direct link and D_c symbols are sent through each cross link. Then, after receiving $\mathbf{y}_i(1)$, transmitter i and receiver i can know D_d desired symbols and $(K-1)D_c$ the other user's symbols. This is possible due to the fact that $M \geq D_d + (K-1)D_c$. At the second time slot, each transmitter sends its new D_d symbols and also forwards the other user's symbols to the corresponding receivers. Consequently, we can see that each receiver can decode $2D_d + (K-1)D_c$ symbols during two-time slots, thus achieving $\Gamma_{fb} \geq KD_d + \frac{D_cK(K-1)}{2}$. ■

Remark 4: Suppose there are sufficiently many antennas at each node (e.g., $M \gg D_d + (K-1)D_c$.). Then, from Corollary 1, we can see that the DoF gain due to feedback increases with the number of users. Let Γ_{fb} and Γ_{no} denote the total DoFs when there is feedback and no feedback, respectively. In addition, consider the case where $D_c = D_d = D$. Then, we have

$$\frac{\Gamma_{fb}}{\Gamma_{no}}=\frac{DK(K+1)/2}{DK}=\frac{K+1}{2}$$

and we can see that the DoF gain is proportional to the number of users.

IV. Proof of Theorem 2

Our achievable scheme operates in two time slots. The achievable scheme employs interference alignment when D_c is sufficiently large. For this section, we categorize beamforming vectors for transmitter $i \in \{1, 2, 3\}$ into seven types:

$$\mathbf{V}_i=\left[\begin{array}{cccc}\mathbf{V}_i^{[1]} & \mathbf{V}_i^{[2]} & \cdots & \mathbf{V}_i^{[7]}\end{array}\right],$$

where $\mathbf{V}_i^{[j]}$ is a concatenation of type j beamforming vectors of transmitter i, i.e.,

$$\mathbf{V}_i^{[j]}=\left[\begin{array}{ccc}\mathbf{v}^{[j]}_{i,\sum_{k=1}^{j}d_i^{[k-1]}+1} & \cdots & \mathbf{v}^{[j]}_{i,\sum_{k=1}^{j}d_i^{k}}\end{array}\right],$$

$d_i^{[j]}$ denotes the number of vectors in $\mathbf{V}_i^{[j]}$, and $d_i^{[0]} = 0$. Here, since we consider the symmetric channel, we set $d_i^{[j]} = d^{[j]}$, $\forall i = 1, 2, 3$.

- $\mathbf{v}_{i,k}^{[1]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $\mathbf{H}_{i+1,i}$, i.e., $\mathbf{H}_{i+1,i}\mathbf{v}_{i,k}^{[1]} = 0$, and $\mathbf{H}_{i,i}\mathbf{v}_{i,k}^{[1]} \neq 0$ and $\mathbf{H}_{i+2,i}\mathbf{v}_{i,k}^{[1]} \neq 0$. Note that since rank$(\mathbf{H}_{i+1,i}) = D_c$, the maximum number of beamforming vectors satisfying this condition is $M - D_c$.
- $\mathbf{v}_{i,k}^{[2]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $\mathbf{H}_{i+2,i}$, i.e., $\mathbf{H}_{i+2,i}\mathbf{v}_{i,k}^{[2]} = 0$, and $\mathbf{H}_{i,i}\mathbf{v}_{i,k}^{[2]} \neq 0$ and $\mathbf{H}_{i+1,i}\mathbf{v}_{i,k}^{[2]} \neq 0$.
- $\mathbf{v}_{i,k}^{[3]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $\left[\begin{array}{cc}\mathbf{H}_{i+1,i} & \mathbf{H}_{i+2,i}\end{array}\right]$, i.e., $\mathbf{H}_{i+1,i}\mathbf{v}_{i,k}^{[3]} = 0$ and $\mathbf{H}_{i+2,i}\mathbf{v}_{i,k}^{[3]} = 0$, and $\mathbf{H}_{i,i}\mathbf{v}_{i,k}^{[3]} \neq 0$. Note that this type of beamforming vector exits only when $M \geq 2D_c$. Assuming $M \geq 2D_c$, the maximum number of beamforming vectors satisfying this condition is $M - 2D_c$.
- After determining $\mathbf{V}_i^{[1]}$, $\forall i = 1, 2, 3$, we construct alignment beamforming vectors for each transmitter. We design $\mathbf{V}_i^{[4]}$ to satisfy

  $$\text{span}\left(\mathbf{H}_{i+1,i}\mathbf{V}_i^{[4]}\right)\subseteq\text{span}\left(\mathbf{H}_{i+1,i+2}\mathbf{V}_{i+2}^{I}\right)$$

  where

  $$\mathbf{V}_i^{I}=\left[\begin{array}{cc}\mathbf{V}_i^{[1]} & \mathbf{V}_i^{[4]}\end{array}\right].$$

  To construct feasible $\mathbf{V}_i^{[4]}$, we employ the beamforming scheme in [9], which is proposed for the non-feedback channel (Set $\mathbf{V}_i^{[4]} = V_i^A$ in [9]).
- Consider the case where $M \geq 2D_d$ and $M \leq 2D_c$. Let $\mathbf{V}_i^{[5]} = \left[\begin{array}{cc}\mathbf{V}_i^{[5,1]} & \mathbf{V}_i^{[5,2]}\end{array}\right]$, where

  $$\mathbf{V}_i^{[5,1]}=\left[\begin{array}{ccc}\mathbf{v}^{[5]}_{i,\sum_{j=1}^{4}d^{[j]}+1} & \cdots & \mathbf{v}^{[5]}_{i,\sum_{j=1}^{4}d^{[j]}+d^{[5]}/2}\end{array}\right],$$
  $$\mathbf{V}_i^{[5,2]}=\left[\begin{array}{ccc}\mathbf{v}^{[5]}_{i,\sum_{j=1}^{4}d^{[j]}+d^{[5]}/2} & \cdots & \mathbf{v}^{[5]}_{i,\sum_{j=1}^{5}d^{[j]}}\end{array}\right].$$

  We construct alignment beamforming vectors $\mathbf{V}_i^{[5,1]}$ and $\mathbf{V}_i^{[5,2]}$ such that

  $$\mathbf{H}_{i,i}\mathbf{V}_i^{[5,1]}=\mathbf{H}_{i,i}\mathbf{V}_i^{[5,2]}=0$$
  $$\mathbf{H}_{i+2,i}\mathbf{V}_i^{[5,1]}=\mathbf{H}_{i+2,i+1}\mathbf{V}_{i+1}^{[5,2]},$$

or equivalently,

$$\underbrace{\begin{bmatrix} \mathbf{H}_{i+2,i} & -\mathbf{H}_{i+2,i+1} \\ \mathbf{H}_{i,i} & \mathbf{0}_{M\times M} \\ \mathbf{0}_{M\times M} & \mathbf{H}_{i+1,i+1} \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} \mathbf{V}_i^{[5,1]} \\ \mathbf{V}_{i+1}^{[5,2]} \end{bmatrix} = 0.$$

Since $\mathbf{T}$ is the $3M\times 2M$ matrix whose rank is $M+2D_d$, we can find feasible $\mathbf{V}_i^{[5,1]}$ and $\mathbf{V}_{i+1}^{[5,2]}$, where $d^{[5]} \le 2M-4D_d$. For the case where $M \le 2D_d$ or $M \ge 2D_c$, we set $d^{[5]} = 0$.

- $\mathbf{v}_{i,k}^{[6]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $[\ \mathbf{H}_{i,i} \quad \mathbf{H}_{i+1,i}\ ]$, i.e, $\mathbf{H}_{i,i}\mathbf{v}_{i,k}^{[6]} = 0$ and $\mathbf{H}_{i+1,i}\mathbf{v}_{i,k}^{[6]} = 0$, and $\mathbf{H}_{i+2,i}\mathbf{v}_{i,k}^{[6]} \neq 0$. Note that this type of beamforming vector exits only when $M \ge D_d + D_c$. Assuming $M \ge D_d + D_c$, the maximum number of beamforming vectors satisfying this condition is $M - D_d - D_c$.
- $\mathbf{v}_{i,k}^{[7]}$ denotes the kth beamforming vector for transmitter i which spans the nullspace of $[\ \mathbf{H}_{i,i} \quad \mathbf{H}_{i+2,i}\ ]$, i.e, $\mathbf{H}_{i,i}\mathbf{v}_{i,k}^{[7]} = 0$ and $\mathbf{H}_{i+2,i}\mathbf{v}_{i,k}^{[7]} = 0$, and $\mathbf{H}_{i+1,i}\mathbf{v}_{i,k}^{[7]} \neq 0$.

Notice that $\mathbf{V}_i^{[4]}$ and $\mathbf{V}_i^{[5]}$ are alignment beamforming matrices while the others are zero-forcing beamforming matrices.

Now we explain the proposed scheme. At time slot t, we design the transmitted signal for transmitter $i \in \{1,2,3\}$ as

$$\mathbf{x}_i(t) = \sum_{j=1}^{7} \mathbf{V}_i^{[j]}\mathbf{s}_i^{[j]}(t).$$

where

$$\mathbf{s}_i^{[j]}(1) = \left[\ s_{i,\sum_{k=1}^{j} d^{[k-1]}+1} \quad \cdots \quad s_{i,\sum_{k=1}^{j} d^{k}}\ \right]^T.$$

Here, transmitters send their symbols with independent Gaussian signaling, and beamforming vectors are properly scaled to satisfy the power constraint. As we mentioned above, our achievable scheme operates in two time slots. In the first time slot, transmitter i delivers the symbols $(\mathbf{s}_i^{[1]}(1), \mathbf{s}_i^{[2]}(1), \mathbf{s}_i^{[3]}(1), \mathbf{s}_i^{[4]}(1))$, $(\mathbf{s}_i^{[5,1]}(1), \mathbf{s}_i^{[7]}(1))$, and $(\mathbf{s}_i^{[5,2]}(1), \mathbf{s}_i^{[6]}(1))$ to receivers i, $i+1$, and $i+2$, respectively, where

$$\mathbf{s}_i^{[5,1]}(1) = \left[\ s_{i,\sum_{k=1}^{5} d^{[k-1]}+1} \quad \cdots \quad s_{i,\sum_{k=1}^{5} d^{[k-1]}+d^{[5]}/2}\ \right],$$
$$\mathbf{s}_i^{[5,2]}(1) = \left[\ s_{i,\sum_{k=1}^{5} d^{[k-1]}+d^{[5]}/2+1} \quad \cdots \quad s_{i,\sum_{k=1}^{5} d^{k}}\ \right].$$

Here, although $(\mathbf{s}_i^{[5,1]}(1), \mathbf{s}_i^{[7]}(1))$ and $(\mathbf{s}_i^{[5,2]}(1), \mathbf{s}_i^{[6]}(1))$ are not desired symbols for receivers $i+1$ and $i+2$, using feedback, transmitters $i+1$ and $i+2$ will forward them to receiver i in the next time slot.

To achieve this, we choose $d^{[j]}$ to satisfy the following conditions.

$$d^{[1]} \le M - D_c \tag{2}$$
$$d^{[2]} \le M - D_c \tag{3}$$
$$d^{[3]} \le \max(M - 2D_c, 0) \tag{4}$$
$$d^{[5]} \le \max(2M - 4D_d, 0, \min(M - 2D, 0)) \tag{5}$$
$$d^{[6]} \le \max(M - D_c - D_d, 0) \tag{6}$$
$$d^{[7]} \le \max(M - D_c - D_d, 0) \tag{7}$$
$$d^{[1]} + d^{[2]} + d^{[3]} + d^{[4]} \le D_d \tag{8}$$
$$d^{[1]} + d^{[4]} + d^{[5]} + d^{[6]} \le D_c \tag{9}$$
$$d^{[2]} + d^{[4]} + d^{[5]} + d^{[7]} \le D_c \tag{10}$$
$$2d^{[1]} + 2d^{[2]} + d^{[3]} + 2d^{[4]} + \frac{3}{2}d^{[5]} + d^{[6]} + d^{[7]} \le M \tag{11}$$
$$\sum_{j=1}^{7} d^{[j]} \le M \tag{12}$$

Here, the conditions (2)-(7) are due to the properties of $\mathbf{V}_i^{[j]}$; (8)-(10) are due to the fact that the number of symbols transmitted through a channel is constrained by the rank of the channel matrix; (11) is due to the fact that the number of received symbols at a receiver should be less than or equal to the number of antennas at the receiver; (12) is due to the fact that the number of transmitted symbols from a transmitter should be less than or equal to the number of antennas at the transmitter. Note that, due to the alignment properties of $\mathbf{V}_i^{[4]}$ and $\mathbf{V}_i^{[5]}$, we have

$$\mathrm{rank}\left(\left[\ \mathbf{H}_{i,i+1}\mathbf{V}_{i+1}^{[1]} \quad \mathbf{H}_{i,i+1}\mathbf{V}_{i+1}^{[4]} \quad \mathbf{H}_{i,i+2}\mathbf{V}_{i+2}^{[4]}\ \right]\right)$$
$$= d^{[1]} + d^{[4]},$$
$$\mathrm{rank}\left(\left[\ \mathbf{H}_{i,i+1}\mathbf{V}_{i+1}^{[5]} \quad \mathbf{H}_{i,i+2}\mathbf{V}_{i+2}^{[5]}\ \right]\right) = \frac{3}{2}d^{[5]}.$$

Then, we have

$$\mathrm{rank}\,(\mathbf{A}_1) = \sum_{j=1}^{4} d^{[j]}, \tag{13}$$
$$\mathrm{rank}\,(\mathbf{A}_2) = d^{[1]} + d^{[4]} + d^{[5]} + d^{[6]}, \tag{14}$$
$$\mathrm{rank}\,(\mathbf{A}_3) = d^{[2]} + d^{[4]} + d^{[5]} + d^{[7]}, \tag{15}$$
$$\mathrm{rank}\left(\left[\ \mathbf{A}_2 \quad \mathbf{A}_3\ \right]\right)$$
$$= d^{[1]} + d^{[2]} + d^{[4]} + \frac{3}{2}d^{[5]} + d^{[6]} + d^{[7]}, \tag{16}$$
$$\mathrm{rank}\left(\left[\ \mathbf{A}_1 \quad \mathbf{A}_2 \quad \mathbf{A}_3\ \right]\right)$$
$$= 2d^{[1]} + 2d^{[2]} + d^{[3]} + 2d^{[4]} + \frac{3}{2}d^{[5]} + d^{[6]} + d^{[7]} \tag{17}$$

with probability one, where

$$\mathbf{A}_1 = \mathbf{H}_{i,i}\left[\ \mathbf{V}_i^{[1]} \quad \mathbf{V}_i^{[2]} \quad \mathbf{V}_i^{[3]} \quad \mathbf{V}_i^{[4]}\ \right],$$
$$\mathbf{A}_2 = \mathbf{H}_{i,i+1}\left[\ \mathbf{V}_{i+1}^{[1]} \quad \mathbf{V}_{i+1}^{[4]} \quad \mathbf{V}_{i+1}^{[5]} \quad \mathbf{V}_{i+1}^{[6]}\ \right],$$
$$\mathbf{A}_3 = \mathbf{H}_{i,i+2}\left[\ \mathbf{V}_{i+2}^{[2]} \quad \mathbf{V}_{i+2}^{[4]} \quad \mathbf{V}_{i+2}^{[5]} \quad \mathbf{V}_{i+2}^{[7]}\ \right].$$

Notice that (13)-(17) are due to the facts that $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ are full-rank matrices and all the channel matrices are generic. Thus, by observing $\mathbf{y}_i(1)$, receiver i and transmitter i can decode the desired symbols $(\mathbf{s}_i^{[1]}(1), \mathbf{s}_i^{[2]}(1), \mathbf{s}_i^{[3]}(1), \mathbf{s}_i^{[4]}(1))$ and the other user's symbols $(\mathbf{s}_{i+1}^{[5,2]}(1), \mathbf{s}_{i+1}^{[6]}(1), \mathbf{s}_{i+2}^{[5,1]}(1), \mathbf{s}_{i+2}^{[7]}(1))$ as desired.

Now we consider the proposed scheme in the second time slot. Recall that transmitter i can know the other user's symbols $(\mathbf{s}_{i+1}^{[5,2]}(1), \mathbf{s}_{i+1}^{[6]}(1), \mathbf{s}_{i+2}^{[5,1]}(1), \mathbf{s}_{i+2}^{[7]}(1))$ after receiving $\mathbf{y}_i(1)$. In the second time slot, each transmitter will forward these symbols to the corresponding receivers, i.e., forward $(\mathbf{s}_{i+1}^{[5,2]}(1), \mathbf{s}_{i+1}^{[6]}(1))$ to receiver $i+1$ and $(\mathbf{s}_{i+2}^{[5,1]}(1), \mathbf{s}_{i+2}^{[7]}(1))$ to receiver $i+2$ in addition to its own new symbols. To achieve this, we set the symbols of user i transmitted at time slot 2 as

$$\begin{aligned}
&\mathbf{s}_i^{[j]}(2)\\
&= \begin{bmatrix} s_{i,\sum_{k=1}^{j} d^{[k-1]}+\sum_{l=1}^{7} d^{[l]}+1} & \cdots & s_{i,\sum_{k=1}^{j} d^{k}+\sum_{l=1}^{7} d^{[l]}} \end{bmatrix}^T,\\
&\mathbf{s}_i^{[5]}(2) = \begin{bmatrix} \mathbf{s}_{i+1}^{[5,2]}(1) & \mathbf{s}_{i+2}^{[5,1]}(1) \end{bmatrix}^T,\\
&\mathbf{s}_i^{[6]}(2) = \mathbf{s}_{i+2}^{[7]}(1),\\
&\mathbf{s}_i^{[7]}(2) = \mathbf{s}_{i+1}^{[6]}(1).
\end{aligned}$$

Here, $\mathbf{s}_i^{[1]}(2)$, $\mathbf{s}_i^{[2]}(2)$, $\mathbf{s}_i^{[3]}(2)$, and $\mathbf{s}_i^{[4]}(2)$ are new symbols of user i transmitted at the second time slot. Then, using the same argument as above, receiver i can decode all the symbols ($\mathbf{s}_i^{[1]}(2)$, $\mathbf{s}_i^{[2]}(2)$, $\mathbf{s}_i^{[3]}(2)$, $\mathbf{s}_i^{[4]}(2)$, $\mathbf{s}_i^{[5]}(1)$, $\mathbf{s}_i^{[6]}(1)$, $\mathbf{s}_i^{[7]}(1)$).

In summary, during two time slots, receiver $i \in \{1,2,3\}$ can decode $2d^{[1]}+2d^{[2]}+2d^{[3]}+2d^{[4]}+d^{[5]}+d^{[6]}+d^{[7]}$ desired symbols, thus achieving total DoF:

$$\Gamma_{fb} \geq d^{[1]}+d^{[2]}+d^{[3]}+d^{[4]}+\frac{d^{[5]}+d^{[6]}+d^{[7]}}{2}.$$

Now we analyze the achievable total DoF by determining suitable $d^{[j]}$ $\forall j = 1,2,\ldots,7$ with respect to M, D_d and D_c.

A. Case 1 : when $D_c \leq M \leq 2D_c$

1) $M \geq 2D_d$ and $M \leq D_d + D_c$: In this case, the proposed scheme employs interference alignment. We set[2]

$$\begin{aligned}
d^{[1]} &= M - D_c,\\
d^{[4]} &= D_d + D_c - M,\\
d^{[2]} &= d^{[3]} = d^{[6]} = d^{[7]} = 0,\\
d^{[5]} &= \frac{2M-4D_d}{3},
\end{aligned}$$

which satisfies the conditions (2)-(12). Then, during two time slots, receiver $i \in \{1,2,3\}$ can decode $2d^{[1]}+2d^{[4]}+d^{[5]} = 2D_d + \frac{2M-4D_d}{3}$ symbols, thus achieving the following total DoF:

$$\begin{aligned}
\Gamma_{fb} &\geq 3\left(D_d + \frac{M-2D_d}{3}\right)\\
&= M + D_d. \qquad (18)
\end{aligned}$$

2) When $M \geq 2D_d$ and $M \geq D_d + D_c$: As in the previous case, the proposed scheme involves interference alignment. We set

$$\begin{aligned}
d^{[1]} &= d^{[2]} = \frac{D_d}{2},\\
d^{[3]} &= d^{[4]} = 0,\\
d^{[5]} &= \frac{4D_c-2M}{3},\\
d^{[6]} &= d^{[7]} = M - D_c - D_d,
\end{aligned}$$

which satisfies the conditions (2)-(12). Then, during two time slots, receiver $i \in \{1,2,3\}$ can decode $2d^{[1]}+2d^{[2]}+d^{[5]}+d^{[6]}+d^{[7]}$ symbols, thus achieving the following total DoF:

$$\begin{aligned}
\Gamma_{fb} &\geq 3\left(M - D_c + \frac{2D_c - M}{3}\right)\\
&= 2M - D_c. \qquad (19)
\end{aligned}$$

3) When $M \leq 2D_d$: In this case, we use the non-feedback scheme in [9] and can achieve

$$\Gamma_{fb} \geq \frac{3M}{2} \qquad (20)$$

by setting

$$\begin{aligned}
d^{[1]} &= M - D_c,\\
d^{[4]} &= D_c - \frac{M}{2},\\
d^{[2]} &= d^{[3]} = d^{[5]} = d^{[6]} = d^{[7]} = 0.
\end{aligned}$$

B. Case 2 : when $2D_c \leq M \leq 2D_c + D_d$

1) When $M \geq D_c + D_d$: In this case, the proposed scheme is merely based on zero forcing ($d^{[4]} = d^{[5]} = 0$). We set

$$\begin{aligned}
d^{[1]} &= d^{[2]} = \frac{2D_c + D_d - M}{2},\\
d^{[3]} &= M - 2D_c,\\
d^{[4]} &= d^{[5]} = 0,\\
d^{[6]} &= d^{[7]} = M - D_c - D_d
\end{aligned}$$

which satisfies the conditions (2)-(12). Then, the achievable total DoF is given by

$$\begin{aligned}
\Gamma_{fb} &\geq 3\left(D_d + M - D_c - D_d\right)\\
&= 3M - 3D_c. \qquad (21)
\end{aligned}$$

2) When $M \leq D_d + D_c$: In this case, we use the non-feedback scheme in [9] and can achieve

$$\Gamma_{fb} \geq 3M - 3D_c \qquad (22)$$

by setting

$$\begin{aligned}
d^{[1]} &= d^{[2]} = \frac{D_c}{2},\\
d^{[3]} &= M - 2D_c,\\
d^{[4]} &= d^{[5]} = d^{[6]} = d^{[7]} = 0.
\end{aligned}$$

[2] If $\frac{2M-4D_d}{3}$ is not an integer, we consider a three-time symbol extension as in [10], [9]. Furthermore, whenever $d^{[j]}$ is not an integer, we can consider a proper symbol extension.

C. Case 3 : when $M \geq 2D_c + D_d$

In this case, we use only $2D_c + D_d$ antennas out of M antennas at each node. Then, from the result in Case 2, we can achieve

$$\Gamma_{fb} \geq 3D_d + 3D_c, \tag{23}$$

by setting

$$\begin{aligned} d^{[1]} &= d^{[2]} = d^{[4]} = d^{[5]} = 0 \\ d^{[3]} &= D_d \\ d^{[6]} &= d^{[7]} = D_c. \end{aligned}$$

V. Proof of Theorem 3

Let $\bar{W}_i \triangleq \{W_{i+1}, W_{i+2}, \ldots, W_K\}$, $\bar{X}_i \triangleq \{\mathbf{x}^n_{i+1}, \mathbf{x}^n_{i+2}, \ldots, \mathbf{x}^n_K\}$, and $\bar{Y}_i \triangleq \{\mathbf{y}^n_{i+1}, \mathbf{y}^n_{i+2}, \ldots, \mathbf{y}^n_K\}$ $\forall i = 1, 2, \ldots, K$, where $\bar{W}_K = \bar{X}_K = \bar{Y}_K = \emptyset$. Starting from Fano's inequality, we have

$$\begin{aligned}
& n\left(\sum_{i=1}^K R_i - \epsilon_n\right) \\
& \leq \sum_{i=1}^K I\left(W_i; \mathbf{y}_i^n\right) \\
& \overset{(a)}{\leq} \sum_{i=1}^K I\left(W_i; \mathbf{y}_i^n, \bar{W}_i, \bar{Y}_i\right) \\
& \overset{(b)}{=} \sum_{i=1}^K I\left(W_i; \mathbf{y}_i^n, \bar{Y}_i | \bar{W}_i\right) \\
& = \sum_{i=1}^K h\left(\mathbf{y}_i^n, \bar{Y}_i | \bar{W}_i\right) - h\left(\mathbf{y}_i^n, \bar{Y}_i | \bar{W}_i, W_i\right) \\
& = \sum_{i=1}^K h\left(\mathbf{y}_i^n | \bar{Y}_i, \bar{W}_i\right) + \sum_{i=1}^K h\left(\bar{Y}_i | \bar{W}_i\right) - h\left(\mathbf{y}_i^n, \bar{Y}_i | \bar{W}_i, W_i\right) \\
& = \sum_{i=1}^K h\left(\mathbf{y}_i^n | \bar{Y}_i, \bar{W}_i\right) - h\left(\mathbf{y}_1^n, \mathbf{y}_2^n, \ldots, \mathbf{y}_K^n | W_1, W_2, \ldots, W_K\right) \\
& + \sum_{i=2}^K h\left(\bar{Y}_i | \bar{W}_i\right) - h\left(\mathbf{y}_i^n, \bar{Y}_i | \bar{W}_i, W_i\right) \\
& + h\left(\mathbf{y}_2^n, \ldots, \mathbf{y}_K^n | W_2, \ldots, W_K\right) \\
& \overset{(c)}{=} \sum_{i=1}^K h\left(\mathbf{y}_i^n | \bar{Y}_i, \bar{W}_i\right) - h\left(\mathbf{y}_1^n, \mathbf{y}_2^n, \ldots, \mathbf{y}_K^n | W_1, W_2, \ldots, W_K\right) \\
& \overset{(d)}{=} \sum_{i=1}^K h\left(\mathbf{y}_i^n \middle| \bar{Y}_i, \bar{W}_i, \bar{X}_i\right) - \sum_{i=1}^K \sum_{t=1}^n h\left(\mathbf{z}_i(t)\right) \\
& = \sum_{i=1}^K h\left(\sum_{j=1}^i \mathbf{H}_{i,j}\mathbf{x}_j^n + \mathbf{z}_i^n \middle| \bar{Y}_i, \bar{W}_i, \bar{X}_i\right) - \sum_{i=1}^K \sum_{t=1}^n h\left(\mathbf{z}_i(t)\right) \\
& \overset{(e)}{\leq} \sum_{i=1}^K \sum_{t=1}^n h\left(\sum_{j=1}^i \mathbf{H}_{i,j}\mathbf{x}_j(t) + \mathbf{z}_i(t)\right) - h\left(\mathbf{z}_i(t)\right)
\end{aligned}$$

where (a) follows from the fact that adding information increases mutual information (providing a genie); (b) follows from the independence of $(W_1, W_2, \ldots, W_K)$; (c) follows from the recursive properties of $\bar{W}_i$ and $\bar{Y}_i$; (d) follows from the fact that $\mathbf{x}_i(t)$ is a function of $(W_i, \mathbf{y}_i^{t-1})$ and $\mathbf{x}_i^n$ is a function of $(W_i, \mathbf{y}_i^n)$; and (e) follows from the fact that conditioning reduces entropy.

Therefore, we have

$$\begin{aligned}
\sum_{i=1}^K R_i &\leq \sum_{i=1}^K h\left(\sum_{j=1}^i \mathbf{H}_{i,j}\mathbf{x}_j + \mathbf{z}_i\right) - h\left(\mathbf{z}_i\right) \\
&\overset{(a)}{\leq} \sum_{i=1}^K \left((D_d + D_c(i-1))\log P + o(\log P)\right) \\
&- \sum_{i=1}^K h\left(\mathbf{z}_i\right) \\
&= \left(D_d K + \frac{K(K-1)D_c}{2}\right)\log P + o(\log P)
\end{aligned}$$

where (a) follows from the fact that the pre-log term of $h(\sum_{j=1}^i \mathbf{H}_{i,j}\mathbf{x}_j + \mathbf{z}_i)$ is constrained by $D_d + D_c(i-1)$. Hence, we get the following upper bound:

$$\Gamma_{fb} \leq \left(D_d K + \frac{K(K-1)D_c}{2}\right).$$

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