Fifty-third Annual Allerton Conference
Allerton House, UIUC, Illinois, USA
September 29 - October 2, 2015

Degrees of Freedom of Bursty Multiple Access Channels with a Relay

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***Abstract*— We investigate the role of relays in multiple access channels (MACs) with bursty user traffic, where intermittent data traffic restricts the users to bursty transmissions. Specifically, we examine a K-user *bursty* MIMO Gaussian MAC with a relay, where bursty traffic of each user is governed by a Bernoulli random process. As our main result, we characterize the degrees of freedom (DoF) region. To this end, we extend noisy network coding, in which relays compress-and-forward, to achieve the DoF cut-set bound. From this result, we establish the necessary and sufficient condition for attaining *collision-free* DoF performances. Also, we show that relays can provide a DoF gain which scales to some extent with additional relay antennas. Our results have practical implications in various scenarios of wireless systems, such as the Internet of Things (IoT) and media access control protocols.**

I. Introduction

Many practical wireless systems can be viewed as multiple access channels (MACs) where multiple transmitters wish to deliver their messages to one receiver. Examples span from an office network in which multiple electronic devices are connected to a Wi-Fi access point (wireless LANs) to a single cell in which many mobile devices communicate with a base station (cellular networks). The standard information-theoretic model that studies these systems is a two-user Gaussian MAC and its capacity region is characterized.

Some work on variants of the MAC has been done. Past work on relay networks developed several coding strategies for the MAC with a relay [1]. Although its capacity region is still unknown, one thing is certain from the derived outer bounds. In the MAC, relays cannot provide a degrees of freedom (DoF) gain [2].

However, it is premature to conclude that relays play little role. Unlike many information-theoretic models which conventionally assume transmissions to occur at all times, in practice, transmissions take place in a bursty manner. One source of such burstiness can be intermittent data traffic that limits the amount of data available for transfer at transmitters. In fact, it is such burstiness that needs particular attention to investigate the role of relays in practical wireless systems.

Hence, in this work, we examine the role of relays in *bursty* MACs. An example in the simplest model, a two-user bursty Gaussian MAC with a relay where all nodes have a single antenna, indicates that employing relays can be beneficial in bursty networks. Fig. 1 demonstrates a scheme

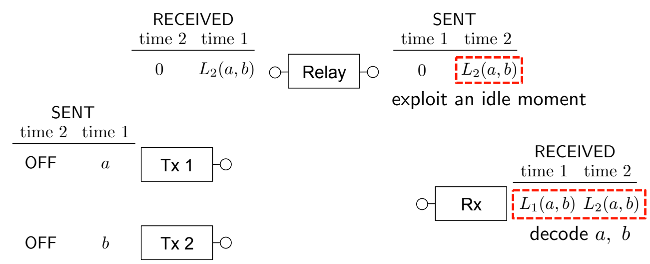


Fig. 1. An achievable scheme in the two-user single-antenna setting. The relay exploits an idle moment of the transmitters to deliver a useful symbol to the receiver. This relay operation helps resolve a collision.

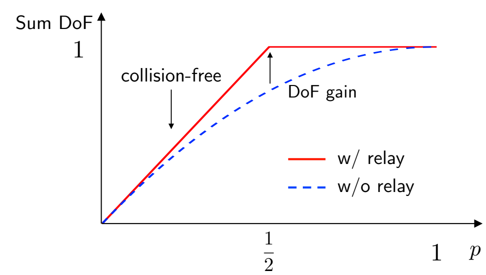


Fig. 2. The sum DoF in the two-user single-antenna setting with and without a relay. We can observe a DoF gain. Interestingly, we can also observe a collision-free DoF performance with low data traffic.

that achieves the sum DoF of the two-user single-antenna setting, where each transmitter is active with probability p. It shows how the relay helps resolve a collision that occurs when both transmitters become active.

- *Time 1:* Both transmitters are active. The receiver gets a linear sum of two symbols. It cannot decode any of them. The relay gets another linear sum of the symbols.
- *Time 2:* Both transmitters are inactive. The relay forwards its past received linear sum. With the two linear sums, the receiver can decode the two symbols.

We see one simple idea: the relay exploits an idle moment of the transmitters. It receives a useful symbol while both transmitters are active, and forwards the symbol to the receiver while they are inactive. This helps resolve a collision.

Using this idea and the DoF cut-set bound, one can readily verify that the sum DoF is $\min(2p, 1)$. Fig. 2 illustrates the sum DoF in comparison with the sum DoF without a relay. We can observe that there is a DoF gain for all levels of data traffic. More interestingly, we can see that with low data traffic ($p < \frac{1}{2}$), both transmitters achieve the individual DoF of p. This is in effect a collision-free DoF performance.

Motivated by this result in the two-user single-antenna setting and the key idea behind it, we further explore a more general setting: a K-user setting where all nodes have multiple antennas. Not only do we aim to characterize the DoF region, we also pay particular attention to practical benefits of employing relays. We raise two questions: "What is the necessary and sufficient condition for attaining *collision-free* DoF performances?" and "Is the DoF gain *scalable* with additional relay antennas?"

The beneficial receive-and-forward relay operation makes it natural for us to extend the noisy network coding scheme [3], which features compress-and-forward strategies of relay nodes [4]. By showing that the extended scheme can achieve the DoF cut-set bound, we characterize the DoF region of the K-user bursty MIMO Gaussian MAC with a relay (Theorem 1). Furthermore, we give answers to the two questions we raise. We establish the necessary and sufficient condition for collision-free DoF performances in the K-user setting: all K transmitters achieve their individual DoF (Corollary 1). Also, we show that relays can offer a DoF gain which scales to some extent with additional relay antennas.

Our results have implications in practical wireless systems, especially where multiple sources deliver data to a single destination in a bursty manner. One such system can be the Internet of Things (IoT), which refers to a network of objects that gather, exchange, and process data. In usual scenarios, many objects gather small amounts of data and deliver them to a central hub, and the hub performs some function based on the collected data. The objects may transmit bursty signals due to intermittent data traffic. Another system can be a network with media access control protocols. Bursty transmissions can take place in such networks, since multiple sources sharing a common medium want to avoid collisions that possibly degrade overall performance. In both systems, our results indicate that relays can be beneficial in achieving higher data throughput, and collision-free communication.

II. Problem Formulation

Fig. 3 describes the K-user *bursty* MIMO Gaussian multiple access channel (MAC) with a relay. The transmitters, the receiver, and the relay have M, N, and L antennas, respectively. Transmitter k wishes to deliver a message W_k reliably to the receiver, $\forall k = 1, \ldots, K$.

Let $X_{kt} \in \mathbb{C}^M$ be transmitter k's encoded signal at time t, and $X_{Rt} \in \mathbb{C}^L$ be the relay's encoded signal at time t. Multiplicative traffic states S_{kt} are assumed to be independent, Bern(p), and i.i.d. over time to govern uncoordinated bursty transmissions[1]. The relay is not restricted to bursty transmissions; it intends to help deliver the messages based on its past received signals, thus it can send signals at all times as long as it has past received signals.

[1]In this work, we consider intermittent data traffic to be a primary source of burstiness. Later in this paper, we discuss random media access control protocols being another source.

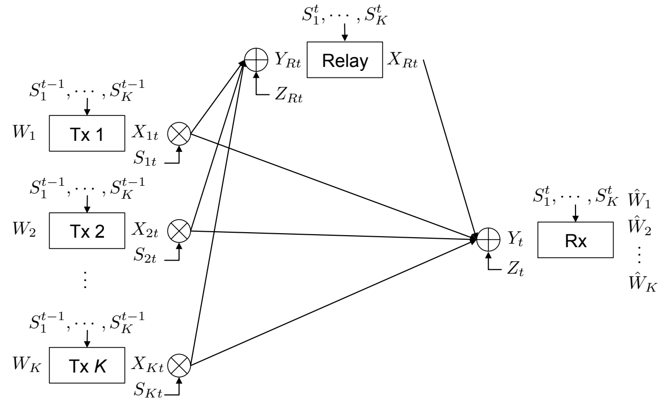


Fig. 3. K-user bursty MIMO Gaussian MAC with a relay.

Additive noise terms Z_t at the receiver and Z_{Rt} at the relay are assumed to be independent, $\mathcal{CN}(0, \mathbf{I}_N)$ and $\mathcal{CN}(0, \mathbf{I}_L)$, and i.i.d. over time. Let $Y_t \in \mathbb{C}^N$ be the received signal of the receiver at time t, and $Y_{Rt} \in \mathbb{C}^L$ be the received signal of the relay at time t.

$$Y_t = \sum_{k=1}^{K} \mathbf{H}_k S_{kt} X_{kt} + \mathbf{H}_R X_{Rt} + Z_t,$$
$$Y_{Rt} = \sum_{k=1}^{K} \mathbf{H}_{Rk} S_{kt} X_{kt} + Z_{Rt}.$$

The matrices $\mathbf{H}_k$ and $\mathbf{H}_{Rk}$ describe the time-invariant channels from transmitter k to the receiver and to the relay respectively. The matrix $\mathbf{H}_R$ describes the time-invariant channel from the relay to the receiver. All channel matrices are assumed to be full rank.

We assume current traffic states are available at the receiver and the relay, since receiving ends can detect which transmitting end is active by measuring the energy levels of incoming signals. We also assume the transmitters get feedback of past traffic states from the receiver. Each transmitter knows its own current traffic state, as it finishes processing the arrivals of data for transmission.

Transmitter k encodes its signal at time t based on its own message, its own current traffic state, and the feedback of past traffic states: $X_{kt} = f_{kt}(W_k, S_{kt}, S^{t-1})$, where S_t stands for $(S_{1t}, \ldots, S_{Kt})$ and S^{t-1} stands for the sequence up to $t-1$. The relay encodes its signal at time t based on its past received signals, and both past and current traffic states: $X_{Rt} = f_{Rt}(Y_R^{t-1}, S^t)$.

We define the DoF region $\mathcal{D} = \{(d_1, \ldots, d_K) : \exists (R_1, \ldots, R_K) \in \mathcal{C}(P)$ such that $d_k = \lim_{P\to\infty} \frac{R_k}{\log P}\}$, where $\mathcal{C}(P)$ is the capacity region with power constraint P.

III. MAIN RESULTS

We characterize the DoF region of the K-user bursty MIMO Gaussian MAC with a relay. We give an outline of the proof in Section IV. The proof in further detail is in Appendix I, where we show that the DoF cut-set bound is achievable by extending noisy network coding [3].

Theorem 1: The DoF region of the K-user bursty MIMO Gaussian MAC with a relay is characterized as follows.

$$\sum_{k\in\mathcal{A}} d_k \le \min\left[\begin{array}{c}\sum_{i=0}^{|A|} P_{|A|,p}(i)\min\left(iM, N+L\right), \\ \sum_{i=0}^{|A|} P_{|A|,p}(i)\min\left(iM+L, N\right)\end{array}\right], \quad (1)$$

where $\mathcal{A} \subseteq \{1,\dots,K\}$ and $P_{|A|,p}(i) := \binom{|A|}{i} p^i(1-p)^{|A|-i}$.

And, we establish the necessary and sufficient condition for attaining collision-free DoF. The proof of necessity is in Appendix II, where we examine if the sum DoF is equal to K times the individual DoF for a certain range of p. The proof of sufficiency follows by Theorem 1.

Corollary 1: The necessary and sufficient condition for attaining collision-free DoF for $p \in (0, p^*)$, where $p^* \in (0,1]$, in the K-user bursty MIMO Gaussian MAC with a relay is as follows.

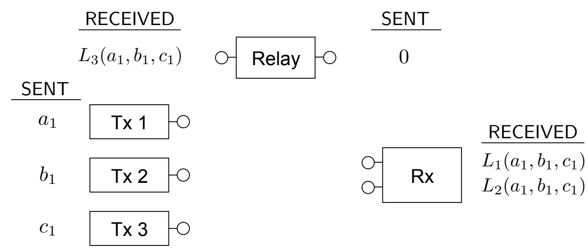
$$KM \le N + L. \quad (2)$$

A. Collision-free DoF performances

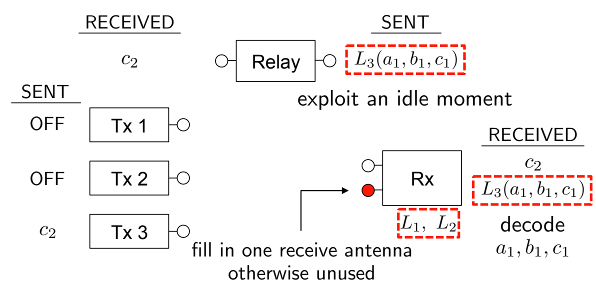
We can answer our first question with Corollary 1. Condition (2) includes an obvious case in which the number of receive antennas is greater than or equal to the total number of transmit antennas ($KM \le N$). In this case, the receiver can decode all symbols instantaneously even when all transmitters become active at the same time. We can achieve the collision-free DoF of KMp without a relay. A relay is of little use.

In the other case ($KM > N$), condition (2) says that we need a relay to achieve collision-free DoF performances and that the relay should have at least $KM - N$ antennas. This is a condition that intuitively comes to mind; when all transmitters become active at the same time, the relay should be able to get the number of linear sums that the receiver additionally needs to decode all symbols.

What is left is to make sense of what operation of the relay makes it possible to achieve collision-free DoF performances. In proving sufficiency for attaining collision-free DoF, we extend the noisy network coding scheme [3], one of whose key ideas is to compress-and-forward [4]. When extending the scheme, we let the relay receive-and-forward without compression. Fig. 4 illustrates the relay operation: when only a few transmitters are active (or none), the relay fills in otherwise unused antennas of the receiver with symbols, the symbols that help resolve past collisions. Again apparent is the key idea in this work: to exploit idle moments of the transmitters. With low data traffic, it is more likely that a few transmitters are active. In such scenarios, compared to the rate at which collisions occur, the relay more



(a) Time 1: $(S_1, S_2, S_3) = (1, 1, 1)$



(b) Time 2: $(S_1, S_2, S_3) = (0, 0, 1)$

Fig. 4. The relay fills in otherwise unused receive antennas with symbols that help resolve collisions. With low data traffic, this happens frequently enough, leading to collision-free DoF performances.

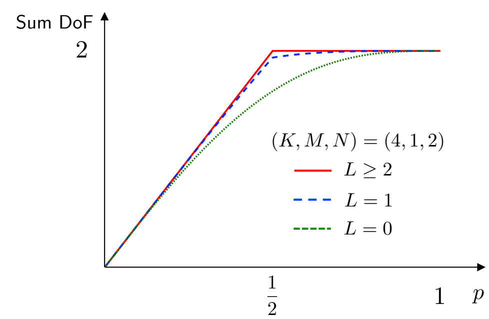


Fig. 5. The sum DoF of three antenna configurations: $(K, M, N, L) = (4, 1, 2, 0)$, $(4, 1, 2, 1)$, $(4, 1, 2, 2)$. We can observe limited scalability of the DoF gain with additional relay antennas.

frequently finds opportunities to deliver symbols intended for resolving the collisions to the receiver, thus leading to collision-free DoF performances.

There is an interesting difference to note between the relay operations in bursty MACs and interference channels (ICs). Recent work on a two-user bursty MIMO Gaussian IC with a relay, focusing on interference-free DoF performances, develops a scheme in which the relay *cooperates* with active transmitters [5]. From this cooperation, they remove interference in the air. This suggests that more sophisticated operations of the relay may be required to achieve optimality in other multi-user bursty networks.

B. DoF gain scalability with relay antennas

Except for the case of $KM \le N$ in which the presence of a relay is of little help, we benefit from having a relay as it provides a DoF gain. To see how much gain it can offer, we compare the sum DoF of three antenna configurations in which only the number of relay antennas varies: $(K, M, N, L) = (4, 1, 2, 0), (4, 1, 2, 1), (4, 1, 2, 2)$. Fig. 5 illustrates the sum DoF of the three antenna configurations. We can answer our second question. We can observe that the DoF gain scales with additional relay antennas. However, the scalability is limited: the gain from adding one additional relay antenna diminishes fast, and soon vanishes.

This limited gain can be explained with an analogy. In bursty MACs, there is one receiver to which all transmitters wish to deliver their message; they are sharing one pie. One transmitter sending at higher rates necessarily means the other transmitters sending at lower rates. Employing a relay is shown to be beneficial, but not significantly. The relay may help the transmitters consume the pie better, but after all, it cannot increase the size of the pie no matter how well it operates.

In bursty ICs, it is a different story. Recently it is shown that a relay can offer a DoF gain in bursty ICs that can scale *linearly* with additional relay antennas [5]. In the IC case, in contrast with the MAC case, each transmitter wishes to deliver their message to its own intended receiver; they are not sharing one pie. One transmitter sending at higher rates could mean the other transmitters sending at lower rates, but it does not result from exclusively consuming the pies of the others. Rather, it results from hindering the others from having theirs. Employing a relay is shown to be significantly beneficial. The relay can help the transmitters consume their own pie only, so that each consumes its own not hindered by the others.

IV. Proof Outline of Theorem 1

In this section, we briefly outline the proof of Theorem 1.

For the outer bound proof, we directly follows the standard cut-set argument. To get the DoF outer bound that matches the claimed DoF region (1), we evaluate the cut-set bound with the Gaussian distributions with power constraint P that maximize the mutual information terms [6]. And we take the limit as $P \to \infty$ after dividing the evaluated cut-set bound by $\log(P)$. Then, we get the DoF outer bound that matches the claimed DoF region (1).

For the inner bound proof, we extend noisy network coding [3]. The transmitters and the relay do not make use of any information of traffic states, although they have access to (part of) it, whereas the receiver does. This is equivalent to the case where information of traffic states is available only at the receiver. Hence, we treat the received signal and the traffic states ((Y, S), where S stands for the traffic states of all transmitters) as the output of the channel. With direct calculations, we get the following achievable rate region.

Lemma 1: An achievable rate region of the K-user bursty MIMO Gaussian MAC with a relay includes the set of $(R_1, \ldots, R_K)$ such that (without time-sharing)

$$\sum_{k\in\mathcal{A}} R_k < \min \begin{bmatrix} I(X(\mathcal{A}); Y, \hat{Y}_R | S, X(\mathcal{A}^c), X_R), \\ I(X(\mathcal{A}), X_R; Y | S, X(\mathcal{A}^c)) \\ -I(Y_R; \hat{Y}_R | S, X_1, \ldots, X_K, X_R, Y) \end{bmatrix}$$

for some distribution $\prod_{k=1}^{K} F(x_k)F(x_R)F(\hat{y}_R|y_R, x_R)$ such that $\mathbb{E}[X_k^2] \leq P$ and $\mathbb{E}[X_R^2] \leq P$, where $\mathcal{A} \subseteq \{1, \ldots, K\}$.

We can compute the rate penalty term (the subtracted mutual information term) for some choice of $\hat{Y}_R$ to show that it does not scale with power constraint P. This is shown in Appendix I. Except for this term, notice that the inner bound is similar to the cut-set bound.

To get the DoF inner bound that matches the claimed DoF region (1), we evaluate the achievable rate region with the independent Gaussian distributions with power constraint P. And we take the limit as $P \to \infty$ after dividing the evaluated achievable rate region by $\log(P)$. Then, the rate penalty term vanishes, and we get the DoF inner bound that matches the claimed DoF region (1).

V. Link to the Internet of Things

The bursty model in this work can be naturally translated into many practical wireless systems. In this section, we discuss implications of our results in one of such systems. As device-to-device communication has been widely available, the Internet of Things (IoT) is receiving attention. A simple example of the IoT can be a network, consisting of a central hub with multiple sensors, that computes the average room temperature: the sensors located at various places deliver measurements to the hub, and the hub computes the average.

Let us consider a natural scenario of the IoT. There is a hub, to which many wireless devices are connected, that wishes to perform some function based on the signals from the devices. It would not be odd to assume that the hub has more antennas than one device has, and less antennas than all devices combined have, since it manages many concurrently. Also, it would be natural to assume each device, constituting a small part of the network, has intermittent traffic of small-sized data to deliver. This scenario well fits with a bursty MAC where $M < N$, $KM > N$, and $p \ll 1$. Here, we can ask the question: if we were to employ a relay to help all devices deliver data at their best, how many antennas should we install at the relay and how should the relay operate?

Our results say that by employing a relay with at least $KM - N$ antennas that performs a simple receive-and-forward operation, we can make all devices connected to the hub deliver their data effectively without collisions. Not only does the relay increase overall data throughput, it also has practical benefits in systems design. When data traffic is sufficiently low, the relay eases the need of the devices to coordinate their transmissions to avoid possible performance degradation. It also eases the need of exchanging acknowledge signals between the hub and the devices to let each other know the receptions of previously sent signals.

VI. Link to Media Access Control Protocols

When we formulate our problem, we consider intermittent data traffic as a primary source of bursty transmissions. They can, however, result from a random media access control protocol with which multiple transmitters sharing a common medium comply. Now, transmitters send signals in a bursty manner, not because data to transfer is intermittently available, but because they want to avoid collisions. Although the source of burstiness is different, our bursty model well captures this scenario.

Let us consider a wireless system with a simple protocol. K transmitters with M transmit antennas wish to send signals to a receiver with N receive antennas. There is a relay with enough antennas from which this wireless system benefits. Similarly as before, suppose $M < N$ and $KM > N$ hold. To avoid collisions that possibly degrade overall performance, each transmitter sends signals according to a protocol: sending signals with probability p and making such decisions independently over time. In this case, probability p no longer represents bursty data traffic as in our original model. Rather, it is now a design parameter of the system. The natural question to ask is: how to choose p to achieve the best performance?

Our results say that by choosing $p = \frac{N}{KM}$, we can achieve the best performance, with no one transmitter lowering its performance for the sake of the others. This threshold probability makes sense, since the relay schedules bursty transmissions of all transmitters and lets them equally share the receiver. The scheduling role of the relay relaxes the complication of the protocol. There is no need of feeding back some information from the receiver to the transmitters to manage collisions.

VII. Conclusion

We characterized the DoF region of the K-user *bursty* MIMO Gaussian MAC with a relay. Moreover, we established the necessary and sufficient condition for achieving *collision-free* DoF performances. Intuitively, the receive-and-forward operation of the relay exploits idle moments of the transmitters, and in effect schedules bursty transmissions to achieve the performances when data traffic is low. We observed that relays can provide a DoF gain which scales to some extent with additional relay antennas. Our results show practical benefits of employing relays into wireless systems where multiple sources wish to deliver data to a single destination in a bursty manner.

Acknowledgment

The authors would like to thank Prof. Song Chong at KAIST for meaningful discussions on practical implications of the results. This work was supported by the National Research Foundation of Korea (NRF) Grant funded by the Korean Government (MSIP) (No. 2015R1C1A1A02036561).

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Appendix I
Proof of Theorem 1

In this appendix, we prove Theorem 1. We give a proof for the two-user setting for simplicity. It is straightforward to extend the proof for the K-user setting.

First, from the standard cut-set argument, we get the following outer bound:

$$\begin{aligned} R_1 &\le \min \begin{bmatrix} I(X_1; Y, Y_R|S, X_2, X_R), \\ I(X_1, X_R; Y|S, X_2) \end{bmatrix}, \\ R_2 &\le \min \begin{bmatrix} I(X_2; Y, Y_R|S, X_1, X_R), \\ I(X_2, X_R; Y|S, X_1) \end{bmatrix}, \\ R_1 + R_2 &\le \min \begin{bmatrix} I(X_1, X_2; Y, Y_R|S, X_R), \\ I(X_1, X_2, X_R; Y|S) \end{bmatrix}, \end{aligned}$$

for all distributions $F(X_1, X_2, X_R)$ such that $\mathbb{E}[X_1^2] \le P$, $\mathbb{E}[X_2^2] \le P$, and $\mathbb{E}[X_R^2] \le P$.

We evaluate the outer bound with the Gaussian distributions with power constraint P that maximize the mutual information terms [6]. And we take the limit as $P \to \infty$ after dividing the evaluated outer bound by $\log(P)$. Then, we get the DoF outer bound that matches (1) for the case of $K = 2$.

Next, we extend the noisy network coding scheme [3]. The transmitters and the relay do not make use of any information of traffic states, although they have access to (part of) it. Hence, transmitter k encodes its signal at time t based on its message: $X_{kt} = f_{kt}(W_k)$. The relay encodes its signal at time t based on its past received signals: $X_{Rt} = f_{Rt}(Y_R^{t-1})$. On the other hand, the receiver makes use of information of traffic states. Thus, we treat (Y_t, S_t) as the output of the channel at time t. Now, we get the following achievable rate region (without time-sharing):

$$\begin{aligned} R_1 &< \min \begin{bmatrix} I(X_1; (Y, S), \hat{Y}_R|X_2, X_R), \\ I(X_1, X_R; (Y, S)|X_2) \\ -I(Y_R; \hat{Y}_R|X_1, X_2, X_R, (Y, S)) \end{bmatrix}, \\ R_2 &< \min \begin{bmatrix} I(X_2; (Y, S), \hat{Y}_R|X_1, X_R), \\ I(X_2, X_R; (Y, S)|X_1) \\ -I(Y_R; \hat{Y}_R|X_1, X_2, X_R, (Y, S)) \end{bmatrix}, \\ R_1 + R_2 &< \min \begin{bmatrix} I(X_1, X_2; (Y, S), \hat{Y}_R|X_R), \\ I(X_1, X_2, X_R; (Y, S)) \\ -I(Y_R; \hat{Y}_R|X_1, X_2, X_R, (Y, S)) \end{bmatrix}, \end{aligned}$$

for some distribution $F(x_1)F(x_2)F(x_R)F(\hat{y}_R|y_R, x_R)$ such that $\mathbb{E}[X_1^2] \le P$, $\mathbb{E}[X_2^2] \le P$, and $\mathbb{E}[X_R^2] \le P$.

The traffic states (S_{1t}, S_{2t}) are independent of the messages (W_1, W_2) and the noise at the relay (Z_R^t). Also, they are i.i.d. over time. Since $X_{kt} = f_{kt}(W_k)$ and $X_{Rt} = f_{Rt}(Y_R^{t-1})$, the traffic states (S_{1t}, S_{2t}) are independent of (X_{1t}, X_{2t}, X_{Rt}). Therefore, $I(X_{k\in\mathcal{B}}; S|X_{k\in\mathcal{B}^c}) = 0$, where $\mathcal{B} \subseteq \{1, 2, R\}$. Using the chain rule and this equality, we calculate the mutual information terms and get the following achievable rate region:

$$R_1 < \min\left[\begin{array}{l} I(X_1;Y,\hat{Y}_R|S,X_2,X_R), \\ I(X_1,X_R;Y|S,X_2) \\ -I(Y_R;\hat{Y}_R|S,X_1,X_2,X_R,Y) \end{array}\right],$$

$$R_2 < \min\left[\begin{array}{l} I(X_2;Y,\hat{Y}_R|S,X_1,X_R), \\ I(X_2,X_R;Y|S,X_1) \\ -I(Y_R;\hat{Y}_R|S,X_1,X_2,X_R,Y) \end{array}\right],$$

$$R_1+R_2 < \min\left[\begin{array}{l} I(X_1,X_2;Y,\hat{Y}_R|S,X_R), \\ I(X_1,X_2,X_R;Y|S) \\ -I(Y_R;\hat{Y}_R|S,X_1,X_2,X_R,Y) \end{array}\right],$$

for some distribution $F(x_1)F(x_2)F(x_R)F(\hat{y}_R|y_R,x_R)$ such that $\mathbb{E}[X_1^2] \le P$, $\mathbb{E}[X_2^2] \le P$, and $\mathbb{E}[X_R^2] \le P$.

We compute the rate penalty term using almost the same method in [6]. We set $\hat{Y}_R = Y_R + \hat{Z}_R$, where $\hat{Z}_R \sim \mathcal{CN}(0, \mathbf{I}_L)$ and is independent of $(S, X_1, X_2, X_R, Y, Y_R)$. We get the following rate penalty:

$$\begin{aligned} & I(Y_R;\hat{Y}_R|S,X_1,X_2,X_R,Y) \\ & \overset{(a)}{=} h(\hat{Y}_R|S,X_1,X_2,X_R,Y) - h(\hat{Y}_R|S,X_1,X_2,X_R,Y,Y_R) \\ & \overset{(b)}{\le} h(\hat{Y}_R|S,X_1,X_2,X_R) - h(\hat{Y}_R|S,X_1,X_2,X_R,Y,Y_R) \\ & \overset{(c)}{=} h(Z_R+\hat{Z}_R) - h(\hat{Z}_R) \overset{(d)}{=} L, \end{aligned}$$

where (a) follows by the chain rule; (b) follows by the fact that conditioning reduces differential entropy; (c) follows by the fact that $\hat{Z}_R$ is independent of $(S, X_1, X_2, X_R, Y, Y_R)$; (d) follows by the fact that $Z_R \sim \mathcal{CN}(0, \mathbf{I}_L)$ and $\hat{Z}_R \sim \mathcal{CN}(0, \mathbf{I}_L)$ are independent.

We evaluate the inner bound with the independent Gaussian distributions with power constraint P. And we take the limit as $P \to \infty$ after dividing the evaluated inner bound by $\log(P)$. Then, the rate penalty term vanishes, and we get the DoF inner bound that matches (1) for the case of $K = 2$.

In conclusion, we get the matching DoF inner and outer bounds. Therefore, we characterize the DoF region of the two-user bursty Gaussian MAC with a relay.

It is straightforward to extend the proof for the two-user setting to that for the K-user setting. The outer bound can be derived from the standard cut-set argument. The inner bound can be derived from the noisy network coding scheme. Except for the fact that the number of input distributions increases, the exact same line of reasoning holds. We characterize the DoF region of the K-user bursty Gaussian MAC with a relay.

$$\sum_{k\in\mathcal{A}} d_k \le \min\left[\begin{array}{l} \sum_{i=0}^{|A|} P_{|A|,p}(i)\min(iM, N+L), \\ \sum_{i=0}^{|A|} P_{|A|,p}(i)\min(iM+L, N) \end{array}\right],$$

where $\mathcal{A} \subseteq \{1,\dots,K\}$ and $P_{|A|,p}(i) := \binom{|A|}{i}p^i(1-p)^{|A|-i}$.

Appendix II
Proof of Corollary 1

In this appendix, we prove Corollary 1. We examine if for a certain class of antenna configurations, an upper bound on the sum DoF is strictly less than K times the individual DoF for all $p \in (0,1)$. Then, the corresponding class is not a necessary condition for attaining collision-free DoF.

If for a certain class of antenna configurations, K times the individual DoF is less than or equal to the sum DoF for $p \in \mathcal{I}$ where $\mathcal{I} \subseteq (0,1)$, then the corresponding class is the necessary and sufficient condition for attaining collision-free DoF, since the individual DoF and the sum DoF are achievable from Theorem 1.

A. $KM > N + L$ and $M \le N$

From $M \le N$, $p\min(M, N+L) = pM$. From $M \le M + L$ and $M \le N$, $pM \le p\min(M+L, N)$. Thus, we get the individual DoF of pM.

Using the fact that $\min(a,b) \le a$, we get an upper bound on the sum DoF: $\sum_{i=0}^{K} P_{K,p}(i)\min(iM, N+L)$.

$$\begin{aligned} & \sum_{i=0}^{K} P_{K,p}(i)\min(iM, N+L) \\ & = \sum_{i=0}^{K-1} P_{K,p}(i)(iM) + P_{K,p}(K)(N+L) \\ & < \sum_{i=0}^{K-1} P_{K,p}(i)(iM) + P_{K,p}(K)(KM) \\ & = \sum_{i=0}^{K} P_{K,p}(i)(iM) = K(pM), \end{aligned}$$

where the last equality is the expectation of a binomial random variable with parameters K and p.

In summary, the upper bound on the sum DoF is strictly less than K times the individual DoF for all $p \in (0,1)$. This class of antenna configurations is not a necessary condition for attaining collision-free DoF.

B. $KM > N + L$, $M > N + L$, and $L = 0$

From $M > N$ and $L = 0$, we get the individual DoF of pN.

Using the fact that $\min(a,b) \le a$ and $L = 0$, we get an upper bound on the sum DoF: $\sum_{i=0}^{K} P_{K,p}(i)\min(iM, N) = \{1-(1-p)^K\}N$.

Let $f(p) = K(pN) - \{1-(1-p)^K\}N$. Since $f(0) = 0$ and $f'(p) = KN\{1-(1-p)^{K-1}\} > 0$ for all $p \in (0,1)$, $f(p) > 0$ for all $p \in (0,1)$.

In summary, the upper bound on the sum DoF is strictly less than K times the individual DoF for all $p \in (0,1)$. This class of antenna configurations is not a necessary condition for attaining collision-free DoF.

C. $KM > N + L$, $M > N + L$, and $L \geq 1$

From $M > N + L$, $p \min(M, N + L) = p(N + L)$. From $M > N$, $p \min(M + L, N) = pN$. Thus, we get the following individual DoF.

$$d \leq \min \{p(N + L), pN + (1 - p) \min(L, N)\}.$$

Let $p_i := \frac{\min(L,N)}{L+\min(L,N)}$. For $0 < p < p_i$, the individual DoF is $p(N + L)$. Using the fact that $\min(a, b) \leq a$, we get an upper bound on the sum DoF: $\sum_{i=0}^{K} P_{K,p}(i) \min(iM, N + L) = \{1 - (1 - p)^K\}(N + L)$. Using the same method in the earlier case, we can verify that the upper bound on the sum DoF is strictly less than K times the individual DoF for $0 < p < p_i$.

For $p_i \leq p < 1$, the individual DoF is $pN + (1 - p) \min(L, N)$. Using the fact that $\min(a, b) \leq b$, we get an upper bound on the sum DoF: $\sum_{i=0}^{K} P_{K,p}(i) \min(iM + L, N) = (1 - p)^K \min(L, N) + \{1 - (1 - p)^K\}N$. For $p_i \leq p < 1$, $(1 - p)^K \min(L, N)$ is strictly less than $K\{(1 - p) \min(L, N)\}$ and so is $\{1 - (1 - p)^K\}N$ than $K(pN)$. In other words, the upper bound on the sum DoF is strictly less than K times the individual DoF for $p_i \leq p < 1$.

In summary, the upper bounds on the sum DoF are strictly less than K times the individual DoF for all $p \in (0, 1)$. This class of antenna configurations is not a necessary condition for attaining collision-free DoF.

D. $KM > N + L$, $N < M \leq N + L$, and $L \geq 1$

From $M \leq N + L$, $p \min(M, N + L) = pM$. From $N < M$, $p \min(M + L, N) = pN$. Thus, we get the individual DoF of $\min \{pM, pN + (1 - p) \min(L, N)\}$. When pM is active, by using the same method in Appendix II-A, when $pN + (1-p) \min(L, N)$ is active, by using the same method in Appendix II-C, we can verify that an upper bound on the sum DoF is strictly less than K times the individual DoF for all $p \in (0, 1)$. This class of antenna configurations is not a necessary condition for attaining collision-free DoF.

E. $KM \leq N$

From $M < N$, we get the individual DoF of pM.

From $KM \leq N$, $\min(iM, N + L) = iM$ and $iM \leq \min(iM + L, N)$ for all integers $i \leq K$. Thus, we get the sum DoF of $\sum_{i=1}^{K} P_{K,p}(i) iM = K(pM)$.

This class of antenna configurations is the necessary and sufficient condition for attaining collision-free DoF for all $p \in (0, 1)$.

F. $N < KM \leq N + L$ and $L \geq 1$

From $M < N + L$, $p \min(M, N + L) = pM$. Thus, we get the following individual DoF.

$$d \leq \min \{pM, p \min(M + L, N) + (1 - p) \min(L, N)\}.$$

When $M \leq N$, pM is active for all $p \in (0, 1)$. Let $f(p) = pM - \{p \min(M+L, N) + (1-p) \min(L, N)\}$. This function is continuous. When $M > N$, since $f(0) < 0$ and $f(1) > 0$, by the intermediate value theorem, there always exists $p_i \in (0, 1)$ such that $f(p_i) = 0$. Thus, for $0 < p < p_i$, pM is active.

From $KM \leq N + L$, $\min(iM, N + L) = iM$ for all non-negative integers $i \leq K$. Thus, we get the following sum DoF.

$$\sum_{k=1}^{K} d_k \leq \min \left[K(pM), \sum_{i=0}^{K} P_{K,p}(i) \min(iM + L, N) \right].$$

Let $f(p) = K(pM) - \sum_{i=0}^{K} P_{K,p}(i) \min(iM + L, N)$. This function is continuous. Since $f(0) < 0$ and $f(1) > 0$, by the intermediate value theorem, there always exists $p_s \in (0, 1)$ such that $f(p_s) = 0$. Thus, for $0 < p < p_s$, $K(pM)$ is active.

Suppose $p_i < p_s$. Then, for $p_i < p < p_s$, K times the individual DoF of $K\{p \min(M+L, N)+(1-p) \min(L, N)\}$ is strictly less than the sum DoF of $K(pM)$. This is a contradiction, since both the individual DoF and the sum DoF are achievable. Thus, $p_s \leq p_i$.

When $M \leq N$, for $0 < p < p_s$, K times the individual DoF of pM is less than or equal to the sum DoF of $K(pM)$. For $p_s \leq p < 1$, the sum DoF of $\sum_{i=0}^{K} P_{K,p}(i) \min(iM + L, N)$ is strictly less than K times the individual DoF of pM.

When $M > N$, for $0 < p < p_s$, K times the individual DoF of pM is less than or equal to the sum DoF of $K(pM)$. For $p_s \leq p < p_i$, the sum DoF of $\sum_{i=0}^{K} P_{K,p}(i) \min(iM + L, N)$ is strictly less than K times the individual DoF of pM. For $p_i \leq p < 1$, the sum DoF of $\sum_{i=0}^{K} P_{K,p}(i) \min(iM + L, N) = (1 - p)^K \min(L, N) + \{1 - (1 - p)^K\}N$ is strictly less than K times the individual DoF of $pN + (1 - p) \min(L, N)$.

In summary, this class of antenna configurations is the necessary and sufficient condition for attaining collision-free DoF for $p \in (0, p_s)$ where $p_s \in (0, 1)$.

In conclusion, $KM \leq N + L$ is the necessary and sufficient condition for attaining collision-free DoF for $p \in (0, p^*)$ where $p^* \in (0, 1]$. $p^* = 1$ if and only if $KM \leq N$.